

Your Name:

Your Signature:

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct**.
- Place  a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 17 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	45	
2	35	
3	20	
<b>Total</b>	100	

1. (45 total points) Answer the following unrelated miscellaneous questions.

(a) (10 points) Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)x_2(t) - 2x_1(t) \\ \dot{x}_2(t) &= x_1(t) - x_2(t) - 1.\end{aligned}$$

Find **two** equilibrium points of the nonlinear system. By two equilibrium points I mean:

$$\mathbf{x}_e^{(1)} = \begin{bmatrix} x_{e1}^{(1)} \\ x_{e2}^{(1)} \end{bmatrix}, \text{ and } \mathbf{x}_e^{(2)} = \begin{bmatrix} x_{e1}^{(2)} \\ x_{e2}^{(2)} \end{bmatrix}.$$

(b) (10 points) You are given a matrix  $A$  with the characteristic polynomial

$$\pi_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3)^4 = 0.$$

In other words,  $A$  has three distinct eigenvalues  $\lambda_{1,2,3}$  of different algebraic multiplicity. Given that

$$\dim \mathcal{N}(A - \lambda_2 I) = 2, \quad \dim \mathcal{N}(A - \lambda_3 I) = 2,$$

obtain **all possible Jordan canonical forms** for  $A$ . You have to be clear and precise. Explain your answer.



(c) (10 points) Consider that

$$A = \mathbf{u}\mathbf{v}^\top = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [4 \ 5 \ 6].$$

Note that  $A$  is a rank one matrix.

**Derive**  $e^{At}$  for any  $\mathbf{u}$  and  $\mathbf{v}$  and then compute  $e^{At}$  for the matrix given above and for

$$t = \frac{1}{\mathbf{v}^\top \mathbf{u}} = \frac{1}{32}.$$



(d) (10 points) Is the following system defined by

$$y(t) = (u(t))^{1.1} + u(t + 1)$$

causal or non-causal? Linear or nonlinear? Time-invariant or time-varying? You have to prove your answers. A one-word answer is not enough.



(e) (5 points) The transfer function matrix of the state space system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

can be written as

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

for any MIMO or SISO system. Find the transfer function  $\mathbf{H}(s)$  when

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{C} = [1 \quad 0], \mathbf{D} = [0 \quad 0].$$

Your  $\mathbf{H}(s)$  should be  $\in \mathbb{R}^{1 \times 2}$

2. (35 total points) The state-space representation of a dynamical system is given as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{C} = [2 \quad 1], \mathbf{x}_0 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{D} = 0.$$

- (a) (5 points) By finding the eigenvalues, eigenvectors of the  $\mathbf{A}$  matrix, compute  $e^{\mathbf{A}t}$  via the diagonal transformation. You have to clearly show your work.

- (b) (5 points) Assume that the control input is  $u(t) = 0$ , compute  $x(t)$  and  $y(t)$ . The initial conditions and state-space matrices are given in the problem description.

- (c) (20 points) Assume that the control input is  $u(t) = 1 + 2e^{-2t}$ , compute  $x(t)$  and  $y(t)$ . The initial conditions and state-space matrices are given in the problem description.



- (d) (5 points) Given your answers to the previous question, compute  $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$  as  $t \rightarrow \infty$ . Which state blows up? Also, find  $y(\infty)$ .



3. (20 total points) In this problem, we will study the equilibrium of Susceptible-Infectious-Susceptible (SIS) in epidemics—similar to what we discussed in class. The dynamics of a simplified SIS model can be written as

$$\frac{dS}{dt} = -\frac{\beta SI}{N(t)} + \gamma I \quad (1)$$

$$\frac{dI}{dt} = \frac{\beta SI}{N(t)} - \gamma I \quad (2)$$

where  $S(t)$  is the number of people that are susceptible at time  $t$  and  $I(t)$  is the number of infected people at time  $t$ , where  $N(t)$  is the total number of people which is a **time-varying quantity**.

Assume that the number of people is fixed, that is  $S(t) + I(t) = N(t)$  where  $N(t)$  is the **time-varying population** of the SIS dynamics.

- (a) (10 points) Given the above assumption, reduce the above dynamical system from 2 states  $(S(t), I(t))$  to a dynamic system with only one state  $I(t)$ . You should obtain something like

$$\dot{I}(t) = f(I(t), \beta, N(t), \gamma)$$

where  $f(\cdot)$  is the function that you should determine.

- (b) (5 points) What is the non-trivial (different than zero) **time-varying equilibrium** of the system? In other words, what is  $I_{eq}(t)$ ?

(c) (5 points) The linearized dynamics of  $I(t)$  can be written as:

$$\dot{I}_{lin}(t) = \left. \frac{\partial f(t)}{\partial I(t)} \right|_{I(t)=I_{eq}(t)} \cdot I_{lin}(t).$$

where  $\left. \frac{\partial f(t)}{\partial I(t)} \right|_{I(t)=I_{eq}(t)}$  means “evaluated at  $I(t) = I_{eq}(t)$ ”. In other words, the linearized dynamic system can be written as

$$\dot{x}(t) = \alpha(t) \cdot x(t)$$

where  $x(t)$  is the linearized state  $I_{lin}(t)$ , and  $\alpha(t) = \left. \frac{\partial f(t)}{\partial I(t)} \right|_{I(t)=I_{eq}(t)}$ . Analyze the stability of this equilibrium point and explain what happens as  $t \rightarrow \infty$  as any of these parameters  $\beta, N(t), \gamma$  change.

Does the stability of the linearized system depend on  $N(t)$ ?

