Your Name: Your Signature:

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 7 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	20	
2	25	
3	15	
4	20	
5	20	
Total	100	

1. (20 total points) Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} = TJT^{-1}, \ B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) (15 points) Find $x(n)$ for any n if $u(k) = 2$ and $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

You might find this summation rule useful:

$$\sum_{j=0}^{k-1} j\alpha^{j-1} = \frac{d}{d\alpha}\sum_{j=0}^{k-1}\alpha^j = \frac{d}{d\alpha}\left[\frac{1-\alpha^k}{1-\alpha}\right] = \frac{1-k\alpha^{k-1}+(k-1)\alpha^k}{(1-\alpha)^2}$$

$$\boxed{x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-1-j}Bu(j) = A^k x(0) + \sum_{j=0}^{k-1} A^j Bu(k-1-j) \quad (*)}$$

- First, we can write A as:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} = TJT^{-1}$$

- Find $A^k = TJ^kT^{-1}$, with $J^k = \begin{bmatrix} 0.5^k & k0.5^{k-1} \\ 0 & 0.5^k \end{bmatrix}$, then:

$$x(k) = TJ^kT^{-1}x(0) + T\sum_{j=0}^{k-1} J^jT^{-1}Bu(k-1-j)$$

- $T^{-1}Bu(k-1-j) = 2T^{-1}B = 2 \cdot v = 2\begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ constant, hence:

$$x(k) = \underbrace{TJ^kT^{-1}x(0)}_{=0} + T\left(\sum_{j=0}^{k-1} J^j\right)(2\cdot v) = T\left(\sum_{j=0}^{k-1} J^j\right)(2\cdot v)$$

- Recall that

$$J^k = \begin{bmatrix} 0.5^k & k0.5^{k-1} \\ 0 & 0.5^k \end{bmatrix} \Rightarrow \sum_{j=0}^{k-1} J^j = \sum_{j=0}^{k-1} \begin{bmatrix} 0.5^j & j0.5^{j-1} \\ 0 & 0.5^j \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{k-1} 0.5^j & \sum_{j=0}^{k-1} j0.5^{j-1} \\ 0 & \sum_{j=0}^{k-1} 0.5^j \end{bmatrix}$$

- Hence,

$$x(k) = T\left(\sum_{j=0}^{k-1} J^j\right)(2\cdot v) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sum_{j=0}^{k-1} 0.5^j & \sum_{j=0}^{k-1} j0.5^{j-1} \\ 0 & \sum_{j=0}^{k-1} 0.5^j \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1-0.5^k}{1-0.5} & \frac{1 - k0.5^{k-1} + (k-1)0.5^k}{(1-0.5)^2} \\ 0 & \frac{1-0.5^k}{1-0.5} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{2 - 2k0.5^{k-1} + 2(k-1)0.5^k}{(1-0.5)^2} \\ 4 - 4 \cdot 0.5^k \end{bmatrix} = x(k) = x(n)$$

(b) (5 points) What happens to $x(n)$ as $n \to \infty$?

Finding the limit of

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{2 - 2k0.5^{k-1} + 2(k-1)0.5^k}{(1-0.5)^2} \\ 4 - 4 \cdot 0.5^k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 4 \end{bmatrix} = \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix}$$

2. (25 total points) You are given the following CT LTI system

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Assume that the control input is constant between two sampling instances, i.e.,

$$u(t) = u(kT) =: u(k), \ \textbf{for} \ kT \le t \le (k+1)T, k = 0, 1, \ldots, k_f,$$

where T is the sampling time.

(a) (20 points) We wish to discretize the above continuous time system, and obtain:

$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k).$$

Find the discretized state space matrices via the **two discretization method** we discussed in class. **You should derive these methods**. Recall that the second discretization method provides more accurate approximations.

First method:

- Use the derivative rule:

$$\dot{x}(t) = \lim_{T\to 0} \frac{x(t+T) - x(t)}{T}$$

- You can use this approximation:

$$\frac{x(t+T) - x(t)}{T} = Ax(t) + Bu(t) \Rightarrow x(t+T) = x(t) + ATx(t) + BTu(t)$$

- Hence,

$$x(t+T) = (I + AT)x(t) + BTu(t)$$

- Now, if we compute $x(t)$ and $y(t)$ only at $t = kT$ for $k = 0, 1, \ldots,$ then the dynamical system equation for the discretized, approximate system is:

$$\begin{aligned} x((k+1)T) &= \underbrace{(I + AT)}_{\tilde{A}} x(kT) + \underbrace{BT}_{\tilde{B}} u(kt) \\ y(kT) &= \tilde{C}x(kT) + \tilde{D}u(kT) \end{aligned}$$

Second method:

- Recall the solution to the state-equation:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Setting $t = KT$ in the previous equation, then we can write:

$$x(k) := x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau$$

$$x(k+1) := x((k+1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

- Note that the above equation can be written as:

$$x(k+1) = e^{AT}\left(e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau\right)$$
$$+ \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}Bu(\tau)d\tau$$

- Recall that we're assuming that:

$$u(t) = u(kT) =: u(k) \qquad \text{for } kT \le t \le (k+1)T,\ k = 0,1,\ldots,k_f$$

i.e., the input is constant between two sampling instances

- Look at $x(k)$ and let $\alpha = kT + T - \tau$, then:

$$x(k+1) = e^{AT}x(k) + \left(\int_0^T e^{A\alpha}d\alpha\right)Bu(k).$$

Hence,

$$\tilde{A} = e^{AT}, \tilde{B} = \left(\int_0^T e^{A\alpha}d\alpha\right)B.$$

(b) (5 points) Obtain $\tilde{A}, \tilde{B}$ given that

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -\pi \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T = \texttt{samplingTime} = 1$$

using either of the discretization methods.

- First method:

$$\tilde{A} = I + TA = \begin{bmatrix} 3 & 0 \\ 0 & 1-\pi \end{bmatrix}, \tilde{B} = TB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Second method:

$$\tilde{A} = exp(AT) = \begin{bmatrix} e^2 & 0 \\ 0 & e^{-\pi} \end{bmatrix}, \tilde{B} = \left(\int_0^T e^{A\alpha}d\alpha\right)B = \begin{bmatrix} 3.2 \\ 0 \end{bmatrix}$$

3. (15 total points) Determine the stability of these systems (marginal, asymptotic, unstable). You have to clearly justify your answer.

   (a) (5 points)

$$x(k+1) = \begin{bmatrix} 0.4 & 1 \\ 2 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 10000000 \\ 0 \end{bmatrix} u(k)$$

You have to find the eigenvalues of A. The evalues of A are: $\{-0.42, 2.82\}$, hence A is unstable since one evalue is outside the unit disk. Therefore, this system is unstable.

   (b) (5 points)

$$\dot{x}(t) = T \begin{bmatrix} -0.4 & 1 & 1 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} T^{-1} x(t) + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} u(t)$$

Unstable since 0.1 is an unstable evalue in the Jordan block.

   (c) (5 points)

$$\dot{x}(t) = \begin{bmatrix} -0.4 & 1 & 1 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t)$$

Marginally stable, since $eig(A) = \{-0.4, -0.4, -0.4, 0\}$. Hence, all eigenvalues of A are in the closed LHP, and the evalue on the $j\omega$ axis has Jordan block of size 1. Hence, the system is marginally stable.

4. (20 total points) You are given the following nonlinear dynamical system:

$$\dot{x}_1(t) = x_1(t)\sin(x_2^2(t)) + x_1^2 x_2(t)u(t) \quad (1)$$
$$\dot{x}_2(t) = x_1(t)e^{-x_2(t)} + \sin(u^2(t)) \quad (2)$$
$$y(t) = 2x_1(t)x_2(t) + x_2^2(t) + u(t). \quad (3)$$

(a) (15 points) Obtain the linearized state space representation of the following nonlinear system around $x_e = \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix}$ and $u_e = u^*$. These equilibrium quantities are assumed to be given. You should obtain A, B, C, D for

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \quad \tilde{y}(t) = C\tilde{x}(t) + D\tilde{u}(t).$$

where $\tilde{x}(t) = x(t) - x_e$ and $\tilde{u}(t) = u(t) - u_e$. Note that A, B, C, D will be a function of the x_e and u_e.

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \sin(x_{2e}^2) + 2x_{1e}x_{2e}u_e & x_{1e}^2 u_e + 2x_{1e}x_{2e}\cos(x_{2e}^2) \\ e^{-x_{2e}} & -x_{1e}e^{-x_{2e}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} \quad (4)$$
$$+ \begin{bmatrix} x_{1e}^2 x_{2e} \\ 2u_e\cos(u_e^2) \end{bmatrix} \Delta u(t) \quad (5)$$
$$\Delta\dot{x}(t) = A\Delta x(t) + B\Delta u(t) \quad (6)$$

and

$$\Delta y(t) = \begin{bmatrix} 2x_{e2} & 2x_{e1} + 2x_{e2} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + 1 \cdot \Delta u(t)$$

where $\Delta x(t) = x(t) - x_e$ and $\Delta u(t) = u(t) - u_e$.

(b) (5 points) Given A, B, C, D, determine the stability of the system around this equilibrium point:

$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, u_e = 0.$$

For the given linearization point, we obtain

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \end{bmatrix}, D = 1.$$

The A-matrix is unstable because it's Jordan form is of size 2, which means e^{At} would go to infinity as t goes to infinity. Hence, the above operating point is a unstable operating point.

5. (20 total points) Consider an LTI CT system

$$\dot{x}(t) = \begin{bmatrix} -2+2t & 4 \\ -1 & 2+2t \end{bmatrix} x(t).$$

(a) (15 points) Obtain the state transition matrix $\phi(t, t_0)$ for the above system. To receive full credit, you have to clearly show your steps.

We can write

$$\begin{bmatrix} -2+2t & 4 \\ -1 & 2+2t \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} + 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_1 + \beta(t) A_2.$$

Note that $A_1 A_2 = A_2 A_1$, and

$$A_1^2 = 0.$$

Hence, A_1 is nilpotent of order 2. Hence,

$$e^{A_1(t-t_0)} = I + (t-t_0)A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (t-t_0) \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1-2(t-t_0) & 4(t-t_0) \\ -(t-t_0) & 1+2(t-t_0) \end{bmatrix}.$$

In addition, we can write

$$e^{A_2(t-t_0)} = e^{t^2-t_0^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\phi(t, t_0) = e^{t^2-t_0^2} \begin{bmatrix} 1-2(t-t_0) & 4(t-t_0) \\ -(t-t_0) & 1+2(t-t_0) \end{bmatrix}$$

(b) (5 points) Is this system asymptotically stable?

No; $\lim_{t\to\infty} \phi(t, t_0) = \infty$.