THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 3

Ahmad F. Taha

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The objective of this homework is to test your understanding of the content of Module 3. Due date of the homework is: Sunday, September 17th @ 11:59pm. You have to upload a single PDF with your clear solutions. Sloppy solutions will not be graded.

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1. Determine which of the following sets are vector spaces. Prove your answer.
   (a) The set of natural numbers.
   (b) The set of square diagonal matrices.
   (c) The set of (square) strictly upper diagonal matrices ($a_{i,j} = 0$ for $i \geq j$).
   (d) The set of bounded sequences, i.e., $\{u[k], k = 0, 1, \ldots, ; |u(k)| < \infty\}$.
   (e) The set of bounded functions $u(t)$ on a predefined interval, such that $|u(t)| \leq K$, where K is a positive number.

2. Is the set $\mathcal{S}$ of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $\mathbb{R}^{2\times 2}$?

3. Is $\mathcal{S} = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} ; a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$?

4. Find the null space, range space, determinant, and rank of the following matrices:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix}.$$
   Confirm your answers on MATLAB. Show your code.

5. Assume that $A = TDT^{-1}$, where D is the diagonal matrix.
   (a) Prove by mathematical induction that $A^k = TD^kT^{-1}$.
   (b) Prove that $e^{At} = Te^{Dt}T^{-1}$.

6. For the following dynamical system:
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$
   compute $x(0)$ when $u(t) = 0$ and $x(2) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$.

7. For the same dynamical system in the previous problem, find $x(0)$ when $u(t) = 1$ and $x(2)$ is the zero vector.

8. You are given that $A = \begin{bmatrix} A_1 & I \\ 0 & A_1 \end{bmatrix}$ where A_1 is a square matrix of dimension n, and A is a square matrix of dimension $2n$.
   (a) Find e^{At} in the simplest possible form.
   Hint: If A, B are two matrices that commute, then $e^{(A+B)} = e^A e^B$. Use this hint after writing A as the sum of two matrices.
   (b) Assume now that $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Find e^{At}.

9. A dynamical system is governed by the following state space dynamics:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t).$$

   (a) Find $e^{A(t-t_0)}$.
   (b) Given that $x(1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$, compute $x(t)$ for $t \geq 1$.
   (c) What is $x(5)$?
   (d) Now assume that $x(1) = 0$, and the control input is $u(t) = 1$. Find the initial condition $x(0)$ that would lead to $x(1)$. In other words, assume that your initial condition is now $x(0)$, which you're required to find given that the control drives the system back to zero.
   (e) Confirm your answers on MATLAB. Show your code.

10. Find e^{At} for the following matrices. The expression you obtain should be a closed form one.
   (a) $A = \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, a \neq 0$
   (b) $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}, a + b + c = 0$
   (c) $A = \lambda_1 \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, a \neq 0$
   (d) $A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$

   You can confirm your answers on MATLAB. Show your code.

11. A dynamical system is governed by the following state space dynamics:

$$\dot{x}(t) = \left(\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} + \lambda I_3 \right) x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

   where $a + b + c = 0$. Find $x(0)$ if $u(t) = 2e^{\lambda t}, \forall t \geq 0$, and $x(2) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$.

12. Prove the following results:
   (a) If $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, then $e^{At} = \begin{bmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{bmatrix}$.
   (b) If $A = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$, then $e^{At} = \begin{bmatrix} \cosh(bt) & \sinh(bt) \\ \sinh(bt) & \cosh(bt) \end{bmatrix}$.
   (c) If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, then $e^{At} = e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$.

13. Find the generalized eigenvectors for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$, the Jordan canonical form, as well as the matrix exponential e^{At}.