Credit goes to Sebastian for his homework solutions.

THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 3

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The objective of this homework is to test your understanding of the content of Module 3. Due date of the homework is: Sunday, September 17th @ 11:59pm. You have to upload a single PDF with your clear solutions. Sloppy solutions will not be graded.

1. Determine which of the following sets are vector spaces. Prove your answer.

   (a) The set of natural numbers.

   (b) The set of square diagonal matrices.

   (c) The set of (square) strictly upper diagonal matrices ($a_{i,j} = 0$ for $i \geq j$).

   (d) The set of bounded sequences, i.e., $\{u[k], k = 0, 1, \ldots, ; |u(k)| < \infty\}$.

   (e) The set of bounded functions $u(t)$ on a predefined interval, such that $|u(t)| \leq K$, where K is a positive number.

   Answer:

   (a) The set of natural numbers $\mathbb{N}$ is not a vector space since there exists $x \in \mathbb{N}$ such that for a constant $\alpha \in \mathbb{R}$ where $\alpha < 0$ we have $\alpha x \notin \mathbb{N}$.

   (b) Suppose A and B are two square diagonal matrices. Then $A + B$ is also a square diagonal matrix. Moreover, for a constant $\alpha \in \mathbb{R}$, αA is also a square diagonal matrix. Hence, the set of square diagonal matrices is a vector space. Both justifications are illustrated as follows

   - $A + B$ is equal to

$$\begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{bmatrix} + \begin{bmatrix} b_1 & 0 & \ldots & 0 \\ 0 & b_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & 0 & \ldots & 0 \\ 0 & a_2 + b_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n + a_n \end{bmatrix}.$$

   - αA is equal to

$$\alpha \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{bmatrix} = \begin{bmatrix} \alpha a_1 & 0 & \ldots & 0 \\ 0 & \alpha a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha a_n \end{bmatrix}.$$

   (c) The set of strictly upper diagonal matrices is a vector space because, for A and B that are two strictly upper diagonal matrices, $A + B$ is also a strictly upper diagonal matrix. In addition, for a constant $\alpha \in \mathbb{R}$, αA is also a strictly upper diagonal matrix. Both justifications are illustrated as follows

   - $A + B$ is equal to

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ 0 & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \ldots & a_{1n} + b_{1n} \\ 0 & a_{22} + b_{22} & \ldots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} + b_{nn} \end{bmatrix}.$$

   - αA is equal to

$$\alpha \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \ldots & \alpha a_{1n} \\ 0 & \alpha a_{22} & \ldots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha a_{nn} \end{bmatrix}.$$

(d) Suppose $u_1(k)$ and $u_2(k)$ are two bounded sequences such that $|u_1(k)| \leq K_1$ and $|u_2(k)| \leq K_2$. Adding both sequences yields $|u_1(k) + u_2(k)| \leq |u_1(k)| + |u_2(k)| \leq K_1 + K_2 < \infty$. Moreover, for a constant $\alpha \in \mathbb{R}$, we have $\alpha|u_1(k)| \leq \alpha K_1 < \infty$. Hence, the set of bounded sequences is a vector space.

(e) Suppose $u_1(k)$ and $u_2(k)$ are two bounded functions such that $|u_1(k)| \leq K$ and $|u_2(k)| \leq K$. Then adding both functions yields $|u_1(k) + u_2(k)| \leq |u_1(k)| + |u_2(k)| \leq 2K$. This shows that any $u(k)$ where $u(k) = u_1(k) + u_2(k)$ has the property of $u(k) \leq 2K$, showing that the set of bounded functions on a predefined interval is not a vector space.

2. Is the set $\mathcal{S}$ of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $\mathbb{R}^{2\times 2}$?

Answer: Yes. The reasons are three folds:

(a) The zero matrix in $\mathbb{R}^{2\times 2}$ can be expressed by setting $a = 0$ and $b = 0$.

(b) For two matrices, we have

$$\begin{bmatrix} 2a_1 & b_1 \\ 3a_1+b_1 & 3b_1 \end{bmatrix} + \begin{bmatrix} 2a_2 & b_2 \\ 3a_2+b_2 & 3b_2 \end{bmatrix} = \begin{bmatrix} 2(a_1+a_2) & b_1+b_2 \\ 3(a_1+a_2)+(b_1+b_2) & 3(b_1+b_2) \end{bmatrix},$$

which the right-hand side is in $\mathcal{S}$.

(c) For a constant $\alpha \in \mathbb{R}$, we have

$$\alpha \begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2\alpha a & \alpha b \\ 3\alpha a + \alpha b & 3\alpha b \end{bmatrix},$$

which the right-hand side is in $\mathcal{S}$.

3. Is $\mathcal{S} = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix}; a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$?

Answer: Suppose v_1, $v_2 \in \mathcal{S}$, then

$$v_1 + v_2 = \begin{bmatrix} a_1+2b_1 \\ a_1+1 \\ a_1 \end{bmatrix} + \begin{bmatrix} a_2+2b_2 \\ a_2+1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (a_1+a_2)+2(b_1+b_2) \\ (a_1+a_2)+2 \\ (a_1+a_2) \end{bmatrix}.$$

Since it is apparent that $v_1 + v_2 \notin \mathcal{S}$, then $\mathcal{S}$ is not a subspace.

4. Find the null space, range space, determinant, and rank of the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix}.$$

Confirm your answers on MATLAB. Show your code.

Answer:

(a) The reduced row echelon of A is

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix},$$

whereas its reduced column echelon is

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & \\ 7 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ -1 & -6 & 0 \end{bmatrix}.$$

- The nullspace of A is the solution of the following equations

$$\begin{bmatrix}1 & 0 & -1\\ 0 & -3 & -6\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Leftrightarrow \begin{bmatrix}a-c\\ -3b-6c\\ 0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

  From the above, we have $a = c$ and $b = -2c$. Hence the nullspace of A is

$$\text{Null}(A) = \begin{bmatrix}c\\ -2c\\ c\end{bmatrix} = c\begin{bmatrix}1\\ -2\\ 1\end{bmatrix},$$

  for $c \in \mathbb{R}$.
- From the reduced column echelon form of A, the range of A can immediately be obtained as

$$\text{Range}(A) = a\begin{bmatrix}1\\0\\-1\end{bmatrix} + b\begin{bmatrix}0\\-3\\-6\end{bmatrix} = a\begin{bmatrix}1\\0\\-1\end{bmatrix} + -3b\begin{bmatrix}0\\1\\2\end{bmatrix} = a\begin{bmatrix}1\\0\\-1\end{bmatrix} + \hat{b}\begin{bmatrix}0\\1\\2\end{bmatrix},$$

  for $a, \hat{b} \in \mathbb{R}$.
- Determinant of A can be computed as

$$\text{Det}(A) = 1(45-48) - 2(36-42) + 3(32-35) = -3 + 12 - 9 = 0.$$

- The number of nonzero rows on the reduced row echelon form of A is 2, hence $\text{Rank}(A) = 2$.

(b) The reduced row echelon of B is

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & -1 & -2 & 2\\ 0 & 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 1\end{bmatrix},$$

whereas its reduced column echelon is

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & -1 & -2 & 2\\ 0 & 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\ 2 & -2 & 1.5 & -0.5\\ 4 & -2 & -3 & -1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ -8 & 4 & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

- The nullspace of B is the solution of the following equations

$$\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Leftrightarrow \begin{bmatrix}a-c\\ b+2c\\ d\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

  From the above, we have $a = c$, $b = -2c$, and $d = 0$. Hence the nullspace of B is

$$\text{Null}(B) = \begin{bmatrix}c\\ -2c\\ c\\ 0\end{bmatrix} = c\begin{bmatrix}1\\ -2\\ 1\\ 0\end{bmatrix},$$

  for $c \in \mathbb{R}$.
- From the reduced column echelon form of B, the range of B can immediately be obtained as

$$\text{Range}(B) = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix},$$

  for $a, b, c \in \mathbb{R}$. This shows that B spans $\mathbb{R}^3$.

- The number of nonzero rows on the reduced row echelon form of B corresponds to the rank of B, that is $\text{Rank}(B) = 3$.

(c) The reduced row echelon of C is

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

whereas its reduced column echelon is

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 4 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

- The nullspace of C is the solution of the following equations

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a - c + 2d \\ b + 4c - d \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

  From the above, we have $a = c - 2d$ and $b = -4c + d$. Hence the nullspace of C is

$$\text{Null}(C) = \begin{bmatrix} c - 2d \\ -4c + d \\ c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

  for $c, d \in \mathbb{R}$.

- From the reduced column echelon form of C, range C can immediately be obtained as

$$\text{Range}(C) = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

  for $a, b \in \mathbb{R}$.

- The number of nonzero rows on the reduced row echelon form of C corresponds to the rank of C, that is $\text{Rank}(C) = 2$.

The MATLAB code for this problem is

```
clear all
clc

A = sym([1,2,3;4,5,6;7,8,9]);

disp('Null(A):')
null(A)
disp('Range(A):')
colspace(A)
disp('Det(A):')
det(A)
disp('Rank(A):')
rank(A)

B = sym([1,2,3,4;0,-1,-2,2;0,0,0,1]);

disp('Null(B):')
null(B)
```

```
disp('Range(B):')
colspace(B)
disp('Rank(B):')
rank(B)

C = sym([1,0,-1,2;2,1,2,3;-1,0,1,-2]);

disp('Null(C):')
null(C)
disp('Range(C):')
colspace(C)
disp('Rank(C):')
rank(C)
```

while the corresponding output is

```
Null(A):

ans =

  1
 -2
  1

Range(A):

ans =

[  1, 0]
[  0, 1]
[ -1, 2]

Det(A):

ans =

0

Rank(A):

ans =

     2

Null(B):

ans =

  1
 -2
  1
  0

Range(B):

ans =
```

```
[ 1, 0, 0]
[ 0, 1, 0]
[ 0, 0, 1]

Rank(B):

ans =

     3

Null(C):

ans =

[  1, -2]
[ -4,  1]
[  1,  0]
[  0,  1]

Range(C):

ans =

[  1, 0]
[  0, 1]
[ -1, 0]

Rank(C):

ans =

     2
```

5. Assume that $A = TDT^{-1}$, where D is the diagonal matrix.
   (a) Prove by mathematical induction that $A^k = TD^kT^{-1}$.
   (b) Prove that $e^{At} = Te^{Dt}T^{-1}$.

   Answer:

   (a) Since $A^1 = TD^1T^{-1} = TDT^{-1} = A$, then we just need to prove that the claim holds for $k+1$. Because $T^{-1}T = I$, then

$$
\begin{aligned}
A^kA &= (TD^kT^{-1})(TDT^{-1}) \\
A^{k+1} &= TD^kT^{-1}TDT^{-1} \\
A^{k+1} &= TD^kIDT^{-1} \\
A^{k+1} &= TD^kDT^{-1} \\
A^{k+1} &= TD^{k+1}T^{-1}.
\end{aligned}
$$

■

(b) From the definition of matrix exponential by Taylor series, we have

$$
\begin{aligned}
e^{At} &= \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots \\
e^{At} &= I + TDT^{-1}t + \frac{(TDT^{-1})^2t^2}{2!} + \frac{(TDT^{-1})^3t^3}{3!} + \dots \\
e^{At} &= I + T(Dt)T^{-1} + \frac{(TDT^{-1})(TDT^{-1})t^2}{2!} + \frac{(TDT^{-1})(TDT^{-1})(TDT^{-1})t^3}{3!} + \dots \\
e^{At} &= TT^{-1} + T(Dt)T^{-1} + \frac{T(D^2t^2)T^{-1}}{2!} + \frac{T(D^3t^3)T^{-1}}{3!} + \dots \\
e^{At} &= T(I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \dots)T^{-1} \\
e^{At} &= Te^{Dt}T^{-1}
\end{aligned}
$$

■

6. For the following dynamical system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

compute $x(0)$ when $u(t) = 0$ and $x(2) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$.
Answer: Suppose that the above system represents the dynamic equation of the form $\dot{x}(t) = Ax(t) + Bu(t)$. Then, we should realize that the matrix A is indeed nilpotent for $k = 2$, because

$$A^2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the expression e^{At} is simply

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix}.$$

The solution of the above system with zero input is

$$
\begin{aligned}
x(t) &= e^{At}x(0) \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.
\end{aligned}
$$

At $t = 2$, we have

$$
\begin{aligned}
\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 2(2) & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} x_1(0) \\ 4x_1(0) + x_2(0) \end{bmatrix}.
\end{aligned}
$$

From the above, we get $x_1(0) = 1$ and $x_2(0) = -4$. Thus, $x(0) = \begin{bmatrix} 1 & -4 \end{bmatrix}^\top$.

7. For the same dynamical system in the previous problem, find $x(0)$ when $u(t) = 1$ and $x(2)$ is the zero vector.

Answer: The solution of the above system with nonzero input is

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & 0 \\ 2(t-\tau) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1)d\tau \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} 1 \\ 2(t-\tau) \end{bmatrix} d\tau \\
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} t \\ t^2 \end{bmatrix}
\end{aligned}
$$

Substituting $t = 2$ to the above yields

$$
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ 4x_1(0) + x_2(0) \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} x_1(0) + 2 \\ 4x_1(0) + x_2(0) + 4 \end{bmatrix}.
$$

From the above, we get $x_1(0) = -2$ and $x_2(0) = 4$. Thus, $x(0) = \begin{bmatrix} -2 & 4 \end{bmatrix}^\top$.

8. You are given that $A = \begin{bmatrix} A_1 & I \\ 0 & A_1 \end{bmatrix}$ where A_1 is a square matrix of dimension n, and A is a square matrix of dimension $2n$.

   (a) Find e^{At} in the simplest possible form.
   Hint: If A, B are two matrices that commute, then $e^{(A+B)} = e^A e^B$. Use this hint after writing A as the sum of two matrices.

   (b) Assume now that $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Find e^{At}.

Answer:

(a) Matrix A must be decomposed such that its resulting matrices are commute. Realize that

$$
A = X + Y = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{Y}
$$

where

$$
\begin{aligned}
XY &= \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_1 \\ 0 & 0 \end{bmatrix} \\
YX &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} = \begin{bmatrix} 0 & A_1 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Since

$$
Y^2 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
$$

then we have

$$
e^{Yt} = I + Yt = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} I & It \\ 0 & I \end{bmatrix}.
$$

Finally, e^{At} can now be expressed as

$$
\begin{aligned}
e^{At} &= e^{Xt}e^{Yt} \\
e^{At} &= e^{Xt} \begin{bmatrix} I & It \\ 0 & I \end{bmatrix}, \quad \text{where } X = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}, \quad \text{or} \\
e^{At} &= \begin{bmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_1 t} \end{bmatrix} \begin{bmatrix} I & It \\ 0 & I \end{bmatrix} \\
e^{At} &= \begin{bmatrix} e^{A_1 t} & te^{A_1 t} \\ 0 & e^{A_1 t} \end{bmatrix}.
\end{aligned}
$$

(b) Since

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

then

$$\begin{aligned} e^{A_1 t} &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\alpha t} & te^{\alpha t} \\ 0 & e^{\alpha t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ e^{A_1 t} &= \begin{bmatrix} e^{\alpha t} & (t+2)e^{\alpha t} \\ 0 & e^{\alpha t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ e^{A_1 t} &= \begin{bmatrix} e^{\alpha t} & te^{\alpha t} \\ 0 & e^{\alpha t} \end{bmatrix}. \end{aligned}$$

Substituting the above into the previous result yields

$$e^{At} = \begin{bmatrix} e^{\alpha t} & te^{\alpha t} & te^{\alpha t} & t^2 e^{\alpha t} \\ 0 & e^{\alpha t} & 0 & te^{\alpha t} \\ 0 & 0 & e^{\alpha t} & te^{\alpha t} \\ 0 & 0 & 0 & e^{\alpha t} \end{bmatrix}.$$

9. A dynamical system is governed by the following state space dynamics:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t).$$

(a) Find $e^{A(t-t_0)}$.

(b) Given that $x(1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$, compute $x(t)$ for $t \geq 1$.

(c) What is $x(5)$?

(d) Now assume that $x(1) = 0$, and the control input is $u(t) = 1$. Find the initial condition $x(0)$ that would lead to $x(1)$. In other words, assume that your initial condition is now $x(0)$, which you're required to find given that the control drives the system back to zero.

(e) Confirm your answers on MATLAB. Show your code.

Answer:

(a) Since A is nilpotent for $k = 3$, or $A^3 = 0$, then

$$\begin{aligned} e^{A(t-t_0)} &= I + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} \\ e^{A(t-t_0)} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2(t-t_0) & 0 & 0 \\ 0 & 6(t-t_0) & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12(t-t_0)^2 & 0 & 0 \end{bmatrix} \\ e^{A(t-t_0)} &= \begin{bmatrix} 1 & 0 & 0 \\ 2(t-t_0) & 1 & 0 \\ 6(t-t_0)^2 & 6(t-t_0) & 1 \end{bmatrix}. \end{aligned}$$

(b) If the system starts at $t = 1$ with $x(1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$, then

$$\begin{aligned} x(t) &= e^{A(t-1)} x(1) + \int_1^t e^{A(t-\tau)} Bu(\tau) d\tau \\ x(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 2(t-1) & 1 & 0 \\ 6(t-1)^2 & 6(t-1) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \int_1^t \begin{bmatrix} 1 & 0 & 0 \\ 2(t-\tau) & 1 & 0 \\ 6(t-\tau)^2 & 6(t-\tau) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(\tau) d\tau \\ x(t) &= \begin{bmatrix} 1 \\ 2t-1 \\ 6t^2-6t+1 \end{bmatrix} + \int_1^t \begin{bmatrix} 1 \\ 2(t-\tau) \\ 6(t-\tau)^2 \end{bmatrix} u(\tau) d\tau, \quad \forall t \geq 1. \end{aligned}$$

If $u(t) = 1$, then

$$x(t) = \begin{bmatrix} 1 \\ 2t-1 \\ 6t^2-6t+1 \end{bmatrix} + \begin{bmatrix} t-1 \\ t^2-1 \\ 2t^3-2 \end{bmatrix} = \begin{bmatrix} t \\ t^2+2t-2 \\ 2t^3+6t^2-6t-1 \end{bmatrix}, \quad \forall t \geq 1.$$

(c) To obtain $x(5)$, putting $t = 5$ to the previous result yields

$$x(5) = \begin{bmatrix} 1 \\ 2(5)-1 \\ 6(5)^2-6(5)+1 \end{bmatrix} + \int_1^5 \begin{bmatrix} 1 \\ 2(5-\tau) \\ 6(5-\tau)^2 \end{bmatrix} u(\tau)d\tau$$

$$x(5) = \begin{bmatrix} 1 \\ 9 \\ 121 \end{bmatrix} + \int_1^5 \begin{bmatrix} 1 \\ 10-2\tau \\ 150-60\tau+\tau^2 \end{bmatrix} u(\tau)d\tau.$$

If $u(t) = 1$, then

$$x(5) = \begin{bmatrix} 1 \\ 9 \\ 121 \end{bmatrix} + \begin{bmatrix} 5-1 \\ (5)^2-1 \\ 2(5)^3-2 \end{bmatrix}$$

$$x(5) = \begin{bmatrix} 1 \\ 9 \\ 121 \end{bmatrix} + \begin{bmatrix} 4 \\ 24 \\ 248 \end{bmatrix}$$

$$x(5) = \begin{bmatrix} 5 \\ 33 \\ 369 \end{bmatrix}.$$

(d) Assuming $x(1) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$ and $u(t) = 0$, the closed-form solution of the dynamics that starts from $t = 0$ is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ 6t^2 & 6t & 1 \end{bmatrix} x(0) + \int_0^t \begin{bmatrix} 1 & 0 & 0 \\ 2(t-\tau) & 1 & 0 \\ 6(t-\tau)^2 & 6(t-\tau) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (1)d\tau$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ 6t^2 & 6t & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \int_0^t \begin{bmatrix} 1 \\ 2(t-\tau) \\ 6(t-\tau)^2 \end{bmatrix} d\tau$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ 2tx_1(0)+x_2(0) \\ 6t^2x_1(0)+6tx_2(0)+x_3(0) \end{bmatrix} + \begin{bmatrix} t \\ 2t-t^2 \\ 2t^3 \end{bmatrix}$$

Substituting $t = 1$, we get

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ 2x_1(0)+x_2(0) \\ 6x_1(0)+6x_2(0)+x_3(0) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

From the above we can infer that $x_1(0) = -1$, $x_2(0) = 3$, and $x_3(0) = -10$. Hence, $x(0) = \begin{bmatrix} -1 & 3 & -10 \end{bmatrix}^\top$.

10. Find e^{At} for the following matrices. The expression you obtain should be a closed form one.

(a) $A = \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, a \neq 0$

(b) $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}, a+b+c=0$

(c) $A = \lambda_1 \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, a \neq 0$

(d) $A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$

You can confirm your answers on MATLAB. Show your code.
Answer:

(a) Since

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix} \begin{bmatrix} a & -a \\ a & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$\begin{aligned} e^{At} &= I + At \\ e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & -a \\ a & -a \end{bmatrix} t \\ e^{At} &= \begin{bmatrix} 1+at & -at \\ at & 1-at \end{bmatrix}. \end{aligned}$$

(b) We have

$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} = \begin{bmatrix} a^2+ab+ac & ab+b^2+bc & ac+bc+c^2 \\ a^2+ab+ac & ab+b^2+bc & ac+bc+c^2 \\ a^2+ab+ac & ab+b^2+bc & ac+bc+c^2 \end{bmatrix}.$$

Substituting $c = -a-b$ to the above yields

$$\begin{bmatrix} a^2+ab+a(-a-b) & ab+b^2+b(-a-b) & a(-a-b)+b(-a-b)+(-a-b)^2 \\ a^2+ab+a(-a-b) & ab+b^2+b(-a-b) & a(-a-b)+b(-a-b)+(-a-b)^2 \\ a^2+ab+a(-a-b) & ab+b^2+b(-a-b) & a(-a-b)+b(-a-b)+(-a-b)^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix exponential can now be computed as

$$\begin{aligned} e^{At} &= I + At \\ e^{At} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} t \\ e^{At} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} at & bt & ct \\ at & bt & ct \\ at & bt & ct \end{bmatrix} \\ e^{At} &= \begin{bmatrix} 1+at & bt & ct \\ at & 1+bt & ct \\ at & bt & 1+ct \end{bmatrix} \\ e^{At} &= \begin{bmatrix} 1+at & bt & (-a-b)t \\ at & 1+bt & (-a-b)t \\ at & bt & 1+(-a-b)t \end{bmatrix} \\ e^{At} &= \begin{bmatrix} 1+at & bt & -at-bt \\ at & 1+bt & -at-bt \\ at & bt & 1-at-bt \end{bmatrix}. \end{aligned}$$

(c) Since

$$\left(\lambda_1 \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}\right)\left(\lambda_1 \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$e^{At} = I + At$$
$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\lambda_1 \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}\right)t$$
$$e^{At} = \begin{bmatrix} 1+\lambda_1 at & -\lambda_1 at \\ \lambda_1 at & 1-\lambda_1 at \end{bmatrix}.$$

(d) Because the matrix is already in a Jordan canonical form, then we simply follow the rules of constructing matrix exponential of Jordan canonical from. That is

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}.$$

The MATLAB code for all above subproblems are

```
clear all
clc

syms a b c lbd1 t

A = [a -a; a -a]

disp('e^(At):')
expm(A*t)

A = [a b c; a b c; a b c]
c = -a-b
A = subs(A)

disp('e^(At):')
expm(A*t)

A = lbd1*[a -a; a -a]

disp('e^(At):')
expm(A*t)

A = [lbd1 1 0; 0 lbd1 1; 0 0 lbd1]

disp('e^(At):')
expm(A*t)
```

and the corresponding results are

```
A =

[ a, -a]
[ a, -a]

e^(At):
```

```
ans =

[ a*t + 1,    -a*t]
[     a*t, 1 - a*t]


A =

[ a, b, c]
[ a, b, c]
[ a, b, c]


c =

- a - b


A =

[ a, b, - a - b]
[ a, b, - a - b]
[ a, b, - a - b]

e^(At):

ans =

[ a*t + 1,     b*t,    -t*(a + b)]
[     a*t, b*t + 1,    -t*(a + b)]
[     a*t,     b*t, 1 - b*t - a*t]


A =

[ a*lbd1, -a*lbd1]
[ a*lbd1, -a*lbd1]

e^(At):

ans =

[ a*lbd1*t + 1,    -a*lbd1*t]
[     a*lbd1*t, 1 - a*lbd1*t]


A =

[ lbd1,    1,    0]
[    0, lbd1,    1]
[    0,    0, lbd1]

e^(At):

ans =
```

```
[ exp(lbd1*t), t*exp(lbd1*t), (t^2*exp(lbd1*t))/2]
[           0,   exp(lbd1*t),       t*exp(lbd1*t)]
[           0,             0,         exp(lbd1*t)]
```

11. A dynamical system is governed by the following state space dynamics:

$$\dot{x}(t) = \left(\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} + \lambda I_3 \right) x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

where $a + b + c = 0$. Find $x(0)$ if $u(t) = 2e^{\lambda t}, \forall t \geq 0$, and $x(2) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$.
Answer: To solve this problem, the first step is to determine e^{At}. Realize that

$$A = X + Y = \underbrace{\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{Y},$$

where the matrix pair (X, Y) is commute, because

$$XY = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ \lambda a & \lambda b & \lambda c \\ \lambda a & \lambda b & \lambda c \end{bmatrix}$$
$$YX = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ \lambda a & \lambda b & \lambda c \\ \lambda a & \lambda b & \lambda c \end{bmatrix}.$$

Based on the commutative relation of (X, Y), the fact that $a + b + c = 0$, and the results from Problem 10, e^{At} can be obtained as

$$e^{At} = e^{Xt} e^{Yt}$$
$$e^{At} = \begin{bmatrix} 1 + at & bt & -at - bt \\ at & 1 + bt & -at - bt \\ at & bt & 1 - at - bt \end{bmatrix} \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$
$$e^{At} = \begin{bmatrix} (1 + at)e^{\lambda t} & bte^{\lambda t} & (-at - bt)e^{\lambda t} \\ ate^{\lambda t} & (1 + bt)e^{\lambda t} & (-at - bt)e^{\lambda t} \\ ate^{\lambda t} & bte^{\lambda t} & (1 - at - bt)e^{\lambda t} \end{bmatrix}.$$

The solution of the above system is computed as follows

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
x(t) &= \begin{bmatrix} (1+at)e^{\lambda t} & bte^{\lambda t} & (-at-bt)e^{\lambda t} \\ ate^{\lambda t} & (1+bt)e^{\lambda t} & (-at-bt)e^{\lambda t} \\ ate^{\lambda t} & bte^{\lambda t} & (1-at-bt)e^{\lambda t} \end{bmatrix} x(0) \\
&\quad + \int_0^t \begin{bmatrix} (1+a(t-\tau))e^{\lambda(t-\tau)} & b(t-\tau)e^{\lambda(t-\tau)} & (-a(t-\tau)-b(t-\tau))e^{\lambda(t-\tau)} \\ a(t-\tau)e^{\lambda(t-\tau)} & (1+b(t-\tau))e^{\lambda(t-\tau)} & (-a(t-\tau)-b(t-\tau))e^{\lambda(t-\tau)} \\ a(t-\tau)e^{\lambda(t-\tau)} & b(t-\tau)e^{\lambda(t-\tau)} & (1-a(t-\tau)-b(t-\tau))e^{\lambda(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (2e^{\lambda\tau})d\tau \\
x(t) &= \begin{bmatrix} (1+at)e^{\lambda t} & bte^{\lambda t} & (-at-bt)e^{\lambda t} \\ ate^{\lambda t} & (1+bt)e^{\lambda t} & (-at-bt)e^{\lambda t} \\ ate^{\lambda t} & bte^{\lambda t} & (1-at-bt)e^{\lambda t} \end{bmatrix} x(0) + \int_0^t \begin{bmatrix} 2e^{\lambda t} \\ 2e^{\lambda t} \\ 2e^{\lambda t} \end{bmatrix} d\tau \\
x(t) &= \begin{bmatrix} (1+at)e^{\lambda t} & bte^{\lambda t} & (-at-bt)e^{\lambda t} \\ ate^{\lambda t} & (1+bt)e^{\lambda t} & (-at-bt)e^{\lambda t} \\ ate^{\lambda t} & bte^{\lambda t} & (1-at-bt)e^{\lambda t} \end{bmatrix} x(0) + \begin{bmatrix} 2\tau e^{\lambda t} \\ 2\tau e^{\lambda t} \\ 2\tau e^{\lambda t} \end{bmatrix} \Bigg|_0^t \\
x(t) &= \begin{bmatrix} (1+at)e^{\lambda t} & bte^{\lambda t} & (-at-bt)e^{\lambda t} \\ ate^{\lambda t} & (1+bt)e^{\lambda t} & (-at-bt)e^{\lambda t} \\ ate^{\lambda t} & bte^{\lambda t} & (1-at-bt)e^{\lambda t} \end{bmatrix} x(0) + \begin{bmatrix} 2te^{\lambda t} \\ 2te^{\lambda t} \\ 2te^{\lambda t} \end{bmatrix}.
\end{aligned}
$$

Substituting $t = 2$ and $x(2) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ yields

$$
\begin{aligned}
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} (1+2a)e^{2\lambda} & 2be^{2\lambda} & (-2a-2b)e^{2\lambda} \\ 2ae^{2\lambda} & (1+2b)e^{2\lambda} & (-2a-2b)e^{2\lambda} \\ 2ae^{2\lambda} & 2be^{2\lambda} & (1-2a-2b)e^{2\lambda} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \begin{bmatrix} 4e^{2\lambda} \\ 4e^{2\lambda} \\ 4e^{2\lambda} \end{bmatrix} \\
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} (1+2a)e^{2\lambda}x_1(0) + 2be^{2\lambda}x_2(0) + (-2a-2b)e^{2\lambda}x_3(0) + 4e^{2\lambda} \\ 2ae^{2\lambda}x_1(0) + (1+2b)e^{2\lambda}x_2(0) + (-2a-2b)e^{2\lambda}x_3(0) + 4e^{2\lambda} \\ 2ae^{2\lambda}x_1(0) + 2be^{2\lambda}x_2(0) + (1-2a-2b)e^{2\lambda}x_3(0) + 4e^{2\lambda} \end{bmatrix}.
\end{aligned}
$$

From the above equation, one can obtain $x(0)$ where

$$x_1(0) = x_2(0) = x_3(0) = e^{-2\lambda} - 4,$$

such that

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} e^{-2\lambda} - 4 \\ e^{-2\lambda} - 4 \\ e^{-2\lambda} - 4 \end{bmatrix}.$$

12. Prove the following results:

    (a) If $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, then $e^{At} = \begin{bmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{bmatrix}$.

    (b) If $A = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$, then $e^{At} = \begin{bmatrix} \cosh(bt) & \sinh(bt) \\ \sinh(bt) & \cosh(bt) \end{bmatrix}$.

    (c) If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, then $e^{At} = e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$.

    Answer:

(a) From the definition of e^{At} in the form of Taylor series, we have

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \dots$$
$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & at \\ -at & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} -a^2t^2 & 0 \\ 0 & -a^2t^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & -a^3t^3 \\ a^3t^3 & 0 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} a^4t^4 & 0 \\ 0 & a^4t^4 \end{bmatrix} + \dots$$
$$e^{At} = \begin{bmatrix} 1 - \frac{1}{2!}a^2t^2 + \frac{1}{4!}a^4t^4 + \dots & at - \frac{1}{3!}a^3t^3 + \dots \\ -at + \frac{1}{3!}a^3t^3 - \dots & 1 - \frac{1}{2!}a^2t^2 + \frac{1}{4!}a^4t^4 + \dots \end{bmatrix}$$
$$e^{At} = \begin{bmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{bmatrix}.$$

■

(b) From the definition of e^{At} in the form of Taylor series, we have

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \dots$$
$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & bt \\ bt & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} b^2t^2 & 0 \\ 0 & b^2t^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & b^3t^3 \\ b^3t^3 & 0 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} b^4t^4 & 0 \\ 0 & b^4t^4 \end{bmatrix} + \dots$$
$$e^{At} = \begin{bmatrix} 1 + \frac{1}{2!}b^2t^2 + \frac{1}{4!}b^4t^4 + \dots & bt + \frac{1}{3!}b^3t^3 + \dots \\ bt + \frac{1}{3!}b^3t^3 + \dots & 1 + \frac{1}{2!}b^2t^2 + \frac{1}{4!}b^4t^4 + \dots \end{bmatrix}$$
$$e^{At} = \begin{bmatrix} \cosh(at) & \sinh(at) \\ \sinh(at) & \cosh(at) \end{bmatrix}.$$

■

(c) Realize that

$$A = X + Y = \underbrace{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}}_{Y},$$

where the matrix pair (X, Y) is commute, because

$$XY = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ ab & 0 \end{bmatrix}$$
$$YX = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & ab \\ ab & 0 \end{bmatrix}.$$

Then, the following applies

$$e^{At} = e^{Xt}e^{Yt}$$
$$e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix}\begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$$
$$e^{At} = \begin{bmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt) \end{bmatrix}$$
$$e^{At} = e^{at}\begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}.$$

■

13. Find the generalized eigenvectors for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$, the Jordan canonical form, as well as the matrix exponential e^{At}.

Answer: First, we need to find the eigenvalue of A. That is

$$\begin{gathered}
\mathrm{Det}(A - \lambda I) = 0 \\
\mathrm{Det}\left(\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = 0 \\
\mathrm{Det}\left(\begin{bmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 2 \\ 0 & -1 & 1-\lambda \end{bmatrix} \right) = 0.
\end{gathered}$$

From the above, we get $(1-\lambda)^3 = 0$, which gives $\lambda = 1$. The eigenvector is

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Leftrightarrow \quad v_1 = c \begin{bmatrix} 1 \\ 0 \\ -0.5 \end{bmatrix}, \quad c \in \mathbb{R}$$

Since the eigenvector of λ spans one column vector, there is only one Jordan block, which is of the form

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad e^{Jt} = \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}.$$

Then, to find the other two eigenvectors, we do the following calculations

$$\begin{gathered}
\begin{bmatrix} 1 \\ 0 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix}, \quad \Rightarrow \quad v_2 = \begin{bmatrix} 2 \\ 0.5 \\ 1 \end{bmatrix} \\
\begin{bmatrix} 2 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_3^1 \\ v_3^2 \\ v_3^3 \end{bmatrix}, \quad \Rightarrow \quad v_3 = \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix},
\end{gathered}$$

hence, we have

$$T = \begin{bmatrix} 1 & 2 & 0.5 \\ 0 & 0.5 & 1 \\ -0.5 & 1 & 0 \end{bmatrix} \quad \text{where} \quad T^{-1} = \begin{bmatrix} 0.5333 & -0.2667 & -0.9333 \\ 0.2667 & -0.1333 & 0.5333 \\ -0.1333 & 1.0667 & -0.2667 \end{bmatrix}.$$

Finally, e^{At} can be computed as follows

$$\begin{aligned}
e^{At} &= Te^{Jt}T^{-1} \\
e^{At} &= \begin{bmatrix} 1 & 2 & 0.5 \\ 0 & 0.5 & 1 \\ -0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 0.5333 & -0.2667 & -0.9333 \\ 0.2667 & -0.1333 & 0.5333 \\ -0.1333 & 1.0667 & -0.2667 \end{bmatrix} \\
e^{At} &= \begin{bmatrix} e^t & (t+2)e^t & \frac{1}{2}(t^2+4t+1)e^t \\ 0 & e^t & \frac{1}{2}(t+2)e^t \\ -\frac{1}{2}e^t & -\frac{1}{2}(t-2)e^t & -\frac{1}{2}(t-4)te^t \end{bmatrix} \begin{bmatrix} 0.5333 & -0.2667 & -0.9333 \\ 0.2667 & -0.1333 & 0.5333 \\ -0.1333 & 1.0667 & -0.2667 \end{bmatrix} \\
e^{At} &= \begin{bmatrix} -\frac{1}{15}(t^2-15)e^t & \frac{2}{15}(4t+15)te^t & -\frac{2}{15}t^2e^t \\ -\frac{1}{15}te^t & \frac{1}{15}(8t+15)e^t & -\frac{2}{15}te^t \\ \frac{1}{30}(t-8)te^t & -\frac{1}{15}(4t-17)te^t & \frac{1}{15}(t^2-8t+15)e^t \end{bmatrix}.
\end{aligned}$$