

The objective of this homework is to test your understanding of the content of Module 4. Due date of the homework is: Thursday, October 5th @ 11:59pm.

1. Consider the following time-varying system:

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{t+1} & 0 \\ -\frac{1}{t+1} & 0 \end{bmatrix} x(t).$$

- Find the state transition matrix.
- Now assume that the system starts from an unknown $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, where $x_1(0) = 1$. Also, assume that $x_2(t=4) = -1$. Given this, find $x_2(0)$.
- After obtaining $x_2(0)$ and given $x_1(0) = 1$, find a general form of $x(t)$ starting $x(0)$ as initial conditions.
- What happens to $x_1(t)$ and $x_2(t)$ as $t \rightarrow \infty$?

Answer:

- Realize that

$$A(t) = \begin{bmatrix} -\frac{1}{t+1} & 0 \\ -\frac{1}{t+1} & 0 \end{bmatrix} = -\frac{1}{t+1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \beta_1(t)A_1,$$

where

$$A_1^2 = A_1A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

hence, A_1 is idempotent. Now, we need to evaluate the following integral

$$\int_{t_0}^t \beta_1(\tau) d\tau = \int_{t_0}^t -\frac{1}{\tau+1} d\tau = -\int_{t_0}^t \frac{1}{\tau+1} d\tau = \ln\left(\frac{t_0+1}{t+1}\right).$$

The state transition matrix now can be formulated as

$$\Phi(t, t_0) = e^{\int_{t_0}^t \beta_1(\tau)A_1 d\tau} = e^{\ln\left(\frac{t_0+1}{t+1}\right)A_1} = e^{\gamma A_1},$$

where $\gamma = \ln\left(\frac{t_0+1}{t+1}\right)$ and

$$\begin{aligned} \Phi(t, t_0) &= e^{\gamma A_1} = \sum_{i=0}^{\infty} \frac{(\gamma A_1)^i}{i!} = I + \gamma A_1 + \frac{\gamma^2 A_1^2}{2!} + \frac{\gamma^3 A_1^3}{3!} + \dots \\ &= I + \gamma A_1 + \frac{\gamma^2 A_1}{2!} + \frac{\gamma^3 A_1}{3!} + \dots \\ &= I + A_1 \left(1 + \gamma + \frac{\gamma^2}{2!} + \frac{\gamma^3}{3!} + \dots - 1\right) \\ &= I + A_1 (e^\gamma - 1) \\ &= I + A_1 \left(\frac{t_0+1}{t+1} - 1\right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \left(\frac{t_0+1}{t+1} - 1\right) \\ &= \begin{bmatrix} \frac{t_0+1}{t+1} & 0 \\ \frac{t_0+1}{t+1} - 1 & 1 \end{bmatrix}. \end{aligned}$$

- (b) We know that the solution is given as $x(t) = \Phi(t, t_0)x(t_0)$. Now, for $t = 4$ and $t_0 = 0$ with $x_1(0) = 1$ and $x_2(4) = -1$, we have

$$\begin{aligned} \begin{bmatrix} x_1(4) \\ x_2(4) \end{bmatrix} &= \begin{bmatrix} \frac{1}{4+1} & 0 \\ \frac{1}{4+1} - 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ \begin{bmatrix} x_1(4) \\ -1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_2(0) \end{bmatrix} \\ \begin{bmatrix} x_1(4) \\ -1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{5} \\ -\frac{4}{5} + x_2(0) \end{bmatrix}. \end{aligned}$$

We finally get $x_2(0) = -\frac{1}{5}$.

- (c) The general form of $x(t)$ is

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{t+1} & 0 \\ \frac{1}{t+1} - 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{5} \end{bmatrix} \\ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{t+1} \\ \frac{1}{t+1} - \frac{6}{5} \end{bmatrix} \end{aligned}$$

- (d) As $t \rightarrow \infty$, we get

$$\lim_{x \rightarrow \infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \lim_{x \rightarrow \infty} \frac{1}{t+1} \\ \lim_{x \rightarrow \infty} \left(\frac{1}{t+1} - \frac{6}{5} \right) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{6}{5} \end{bmatrix}.$$

2. The STM of $\dot{x}(t) = A(t)x(t)$ is given as follows:

$$\phi(t, t_0) = e^{t_0^2 - t^2} \begin{bmatrix} 1 & \ln\left(\frac{t+1}{t_0+1}\right) \\ 0 & 1 \end{bmatrix}.$$

- (a) Given any STM, how can you obtain $A(t)$ back? Prove that $A(t) = \dot{\phi}(t, t)$ for any LTV system.
 (b) For the STM given in this problem, obtain $A(t)$.
 (c) Find $\phi^{-1}(t, t_0)$

Answer:

- (a) Proof:

$$\begin{aligned} \dot{\Phi}(t, t_0) &= \frac{d}{dt} \Phi(t, t_0) = \frac{d}{dt} e^{\int_{t_0}^t A(\tau) d\tau} \\ &= \frac{d}{dt} \left(I + \int_{t_0}^t A(\tau) d\tau + \frac{1}{2!} \left(\int_{t_0}^t A(\tau) d\tau \right)^2 + \frac{1}{3!} \left(\int_{t_0}^t A(\tau) d\tau \right)^3 + \dots \right) \\ &= \frac{d}{dt} \int_{t_0}^t A(\tau) d\tau + \left(\int_{t_0}^t A(\tau) d\tau \right) \frac{d}{dt} \int_{t_0}^t A(\tau) d\tau + \frac{1}{2!} \left(\int_{t_0}^t A(\tau) d\tau \right)^2 \frac{d}{dt} \int_{t_0}^t A(\tau) d\tau + \dots \\ &= A(t) + \left(\int_{t_0}^t A(\tau) d\tau \right) A(t) + \frac{1}{2!} \left(\int_{t_0}^t A(\tau) d\tau \right)^2 A(t) + \frac{1}{3!} \left(\int_{t_0}^t A(\tau) d\tau \right)^3 A(t) + \dots, \end{aligned}$$

since $A(t)$ commutes with its integral, then

$$\begin{aligned} \dot{\Phi}(t, t_0) &= A(t) + A(t) \left(\int_{t_0}^t A(\tau) d\tau \right) + \frac{1}{2!} A(t) \left(\int_{t_0}^t A(\tau) d\tau \right)^2 + \frac{1}{3!} A(t) \left(\int_{t_0}^t A(\tau) d\tau \right)^3 + \dots \\ &= A(t) \left(I + \int_{t_0}^t A(\tau) d\tau + \frac{1}{2!} \left(\int_{t_0}^t A(\tau) d\tau \right)^2 + \frac{1}{3!} \left(\int_{t_0}^t A(\tau) d\tau \right)^3 + \dots \right) \\ &= A(t) e^{\int_{t_0}^t A(\tau) d\tau} \\ &= A(t) \Phi(t, t_0), \end{aligned}$$

setting $t_0 = t$, and knowing the fact that $\Phi(t, t) = I$, we get

$$\begin{aligned}\dot{\Phi}(t, t) &= A(t)\Phi(t, t) \\ &= A(t)\end{aligned}$$

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(b) Since

$$\phi(t, t_0) = e^{t_0^2 - t^2} \begin{bmatrix} 1 & \ln\left(\frac{t+1}{t_0+1}\right) \\ 0 & 1 \end{bmatrix},$$

then the derivative of $\phi(t, t_0)$ is

$$\begin{aligned}\dot{\phi}(t, t_0) &= \frac{d}{dt}\Phi(t, t_0) = -2te^{t_0^2 - t^2} \begin{bmatrix} 1 & \ln\left(\frac{t+1}{t_0+1}\right) \\ 0 & 1 \end{bmatrix} + e^{t_0^2 - t^2} \begin{bmatrix} 0 & \frac{1}{t+1} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2te^{t_0^2 - t^2} & -2te^{t_0^2 - t^2} \ln\left(\frac{t+1}{t_0+1}\right) + \frac{e^{t_0^2 - t^2}}{t+1} \\ 0 & -2te^{t_0^2 - t^2} \end{bmatrix}.\end{aligned}$$

Setting $t_0 = t$ yields

$$\dot{\phi}(t, t_0) = \begin{bmatrix} -2t & \frac{1}{t+1} \\ 0 & -2t \end{bmatrix},$$

which implies

$$A(t) = \begin{bmatrix} -2t & \frac{1}{t+1} \\ 0 & -2t \end{bmatrix}.$$

(c) $\phi^{-1}(t, t_0)$ can be computed as

$$\begin{aligned}\phi^{-1}(t, t_0) &= \phi(t_0, t) \\ &= e^{t^2 - t_0^2} \begin{bmatrix} 1 & \ln\left(\frac{t_0+1}{t+1}\right) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{t^2 - t_0^2} & e^{t^2 - t_0^2} \ln\left(\frac{t_0+1}{t+1}\right) \\ 0 & e^{t^2 - t_0^2} \end{bmatrix}.\end{aligned}$$

3. The following system

$$\dot{x}(t) = A(t)x(t) + e^{-\alpha^2(t)} \begin{bmatrix} \pi \\ 0 \end{bmatrix} u(t).$$

The STM for this system is given by:

$$\phi(t, t_0) = e^{\alpha^2(t_0) - \alpha^2(t)} \begin{bmatrix} 1 & 0 \\ \cos(\pi t) - \cos(\pi t_0) & 1 \end{bmatrix}.$$

- Compute $A(t)$.
- Determine the inverse of the STM.
- Find $x(t)$ if $x(t_0) = 0$ and $u(t) = 1$.

Answer:

(a) The derivative of $\phi(t, t_0)$ can be computed as

$$\begin{aligned}\dot{\Phi}(t, t_0) &= \frac{d}{dt}\phi(t, t_0) \\ &= \begin{bmatrix} -2\alpha(t)\dot{\alpha}(t)e^{\alpha^2(t_0)-\alpha^2(t)} & 0 \\ -2\alpha(t)\dot{\alpha}(t)e^{\alpha^2(t_0)-\alpha^2(t)}(\cos(\pi t) - \cos(\pi t_0)) + e^{\alpha^2(t_0)-\alpha^2(t)}(\pi \sin(\pi t)) & -2\alpha(t)\dot{\alpha}(t)e^{\alpha^2(t_0)-\alpha^2(t)} \end{bmatrix},\end{aligned}$$

setting $t_0 = t$, then we get

$$A(t) = \dot{\Phi}(t, t_0) = \begin{bmatrix} -2\alpha(t)\dot{\alpha}(t) & 0 \\ \pi \sin(\pi t) & -2\alpha(t)\dot{\alpha}(t) \end{bmatrix}.$$

(b)

$$\begin{aligned}\phi^{-1}(t, t_0) &= \phi(t_0, t) = e^{\alpha^2(t)-\alpha^2(t_0)} \begin{bmatrix} 1 & 0 \\ \cos(\pi t_0) - \cos(\pi t) & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\alpha^2(t)-\alpha^2(t_0)} & 0 \\ e^{\alpha^2(t)-\alpha^2(t_0)}(\cos(\pi t_0) - \cos(\pi t)) & e^{\alpha^2(t)-\alpha^2(t_0)} \end{bmatrix}.\end{aligned}$$

(c) The solution is given as

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau) d\tau,$$

for $t_0 = 0$ and $x(0) = 0$, then

$$\begin{aligned}x(t) &= \int_0^t \begin{bmatrix} e^{\alpha^2(\tau)-\alpha^2(t)} & 0 \\ e^{\alpha^2(\tau)-\alpha^2(t)}(\cos(\pi t) - \cos(\pi \tau)) & e^{\alpha^2(\tau)-\alpha^2(t)} \end{bmatrix} \begin{bmatrix} \pi e^{-\alpha^2(\tau)} \\ 0 \end{bmatrix} (1) d\tau \\ &= \int_0^t \begin{bmatrix} \pi e^{-\alpha^2(t)} \\ \pi e^{-\alpha^2(t)}(\cos(\pi t) - \cos(\pi \tau)) \end{bmatrix} d\tau \\ &= \left[\begin{bmatrix} \pi \tau e^{-\alpha^2(t)} \\ \pi \tau e^{-\alpha^2(t)} \cos(\pi t) - e^{-\alpha^2(t)} \sin(\pi \tau) \end{bmatrix} \right] \Big|_0^t \\ &= \begin{bmatrix} \pi t e^{-\alpha^2(t)} \\ \pi t e^{-\alpha^2(t)} \cos(\pi t) - e^{-\alpha^2(t)} \sin(\pi t) \end{bmatrix} \\ &= \begin{bmatrix} \pi t e^{-\alpha^2(t)} \\ e^{-\alpha^2(t)}(\pi t \cos(\pi t) - \sin(\pi t)) \end{bmatrix}.\end{aligned}$$

4. Find the STM associated with

$$A(t) = \begin{bmatrix} \sin(t) & \cos(t) & \beta \\ 0 & \sin(t) & \cos(t) \\ 0 & 0 & \sin(t) \end{bmatrix}.$$

Answer:

$$A(t) = \begin{bmatrix} \sin(t) & \cos(t) & \beta \\ 0 & \sin(t) & \cos(t) \\ 0 & 0 & \sin(t) \end{bmatrix} = \sin(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cos(t) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, the followings must be computed

- $\int_{t_0}^t \sin(\tau) d\tau = -\cos(t) + \cos(t_0)$
- $\int_{t_0}^t \cos(\tau) d\tau = \sin(t) - \sin(t_0)$

- $\int_{t_0}^t \beta d\tau = \beta(t - t_0)$

and the exponential matrices

- Since A_1 is identity, then

$$e^{\int_{t_0}^t \sin(\tau) A_1 d\tau} = e^{-\cos(t) + \cos(t_0)} I = \begin{bmatrix} e^{-\cos(t) + \cos(t_0)} & 0 & 0 \\ 0 & e^{-\cos(t) + \cos(t_0)} & 0 \\ 0 & 0 & e^{-\cos(t) + \cos(t_0)} \end{bmatrix}.$$

- Notice that A_2 is nilpotent for $k = 3$, thus

$$\begin{aligned} e^{\int_{t_0}^t \cos(\tau) A_2 d\tau} &= I + A_2(\sin(t) - \sin(t_0)) + \frac{1}{2!} A_2^2 (\sin(t) - \sin(t_0))^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (\sin(t) - \sin(t_0)) + \frac{1}{2!} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\sin(t) - \sin(t_0))^2 \\ &= \begin{bmatrix} 1 & \sin(t) - \sin(t_0) & \frac{1}{2}(\sin(t) - \sin(t_0))^2 \\ 0 & 1 & \sin(t) - \sin(t_0) \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

- Notice that A_3 is nilpotent for $k = 2$, thus

$$\begin{aligned} e^{\int_{t_0}^t \beta A_3 d\tau} &= I + A_3 \beta(t - t_0) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \beta(t - t_0) \\ &= \begin{bmatrix} 1 & 0 & \beta(t - t_0) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Because A_1, A_2 , and A_3 commute, then

$$\begin{aligned} \Phi(t, t_0) &= e^{\int_{t_0}^t A(\tau) d\tau} \\ &= e^{\int_{t_0}^t \sin(\tau) A_1 d\tau} e^{\int_{t_0}^t \cos(\tau) A_2 d\tau} e^{\int_{t_0}^t \beta A_3 d\tau} \\ &= e^{-\cos(t) + \cos(t_0)} \begin{bmatrix} 1 & \sin(t) - \sin(t_0) & \frac{1}{2}(\sin(t) - \sin(t_0))^2 \\ 0 & 1 & \sin(t) - \sin(t_0) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta(t - t_0) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

5. Consider the following dynamical system:

$$\dot{x}(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} x(t).$$

(a) Derive this structure for the STM:

$$\phi(t, t_0) = \begin{bmatrix} \phi_{11}(t, t_0) & \phi_{12}(t, t_0) \\ 0 & \phi_{22}(t, t_0) \end{bmatrix}.$$

You should basically show that $\dot{\phi}_{ii}(t, t_0) = A_{ii}(t)\phi_{ii}(t, t_0)$, and then show an explicit form for $\phi_{12}(t, t_0)$.

Hint: Remember that $\dot{\phi}(t, t_0) = A(t)\phi(t, t_0)$ and $\phi(t_0, t_0) = I$. You should use that to prove the following result:

$$\phi_{12}(t, t_0) = \int_{t_0}^t \phi_{11}(t, \tau) A_{12}(\tau) \phi_{22}(\tau, t_0) d\tau.$$

(b) Assume that the dynamics for a system are given by:

$$\dot{x}(t) = \begin{bmatrix} 0 & 2t \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2t \end{bmatrix} u(t).$$

Use the results you developed in part (a) to determine $x(t)$ giving that the initial conditions for the system are zero, and $u(t) = 1$, without evaluating this integral $\int_{t_0}^t \phi(t, \tau) B(\tau) u(\tau) d\tau$.

Answer:

(a) Since $\dot{\phi}(t, t_0) = A(t)\phi(t, t_0)$, then

$$\begin{aligned} \dot{\phi}(t, t_0) &= A(t)\phi(t, t_0) \\ \dot{\phi}(t, t_0) &= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} \begin{bmatrix} \phi_{11}(t, t_0) & \phi_{12}(t, t_0) \\ 0 & \phi_{22}(t, t_0) \end{bmatrix} \\ \begin{bmatrix} \dot{\phi}_{11}(t, t_0) & \dot{\phi}_{12}(t, t_0) \\ 0 & \dot{\phi}_{22}(t, t_0) \end{bmatrix} &= \begin{bmatrix} A_{11}(t)\phi_{11}(t, t_0) & A_{11}(t)\phi_{12}(t, t_0) + A_{12}(t)\phi_{22}(t, t_0) \\ 0 & A_{22}(t)\phi_{22}(t, t_0) \end{bmatrix}. \end{aligned}$$

From the above, it is apparent that $\dot{\phi}_{ii}(t, t_0) = A_{ii}(t)\phi_{ii}(t, t_0)$. Now, we need to show the explicit form of $\phi_{12}(t, t_0)$. From the above, we also have

$$\dot{\phi}_{12}(t, t_0) = A_{11}(t)\phi_{12}(t, t_0) + A_{12}(t)\phi_{22}(t, t_0).$$

Pre-multiply with $\phi_{11}(t_0, t)$ yields

$$\phi_{11}(t_0, t)\dot{\phi}_{12}(t, t_0) - \phi_{11}(t_0, t)A_{11}(t)\phi_{12}(t, t_0) = \phi_{11}(t_0, t)A_{12}(t)\phi_{22}(t, t_0).$$

Realize that $\dot{\phi}_{11}(t, t_0)$ is equal to

$$\begin{aligned} \dot{\phi}_{11}(t, t_0) &= \frac{d}{dt} e^{\int_t^{t_0} A_{11}(\tau) d\tau} \\ &= \frac{d}{dt} \left(I + \int_t^{t_0} A_{11}(\tau) d\tau + \frac{1}{2!} \left(\int_t^{t_0} A_{11}(\tau) d\tau \right)^2 + \frac{1}{3!} \left(\int_t^{t_0} A_{11}(\tau) d\tau \right)^3 + \dots \right) \\ &= \frac{d}{dt} \int_t^{t_0} A_{11}(\tau) d\tau + \left(\int_t^{t_0} A_{11}(\tau) d\tau \right) \frac{d}{dt} \int_t^{t_0} A_{11}(\tau) d\tau \\ &\quad + \frac{1}{2!} \left(\int_t^{t_0} A_{11}(\tau) d\tau \right)^2 \frac{d}{dt} \int_t^{t_0} A_{11}(\tau) d\tau + \dots \\ &= -A_{11}(t) - \left(\int_t^{t_0} A_{11}(\tau) d\tau \right) A_{11}(t) - \frac{1}{2!} \left(\int_t^{t_0} A_{11}(\tau) d\tau \right)^2 A_{11}(t) \\ &\quad - \frac{1}{3!} \left(\int_t^{t_0} A_{11}(\tau) d\tau \right)^3 A_{11}(t) - \dots \\ &= - \left(I + \int_t^{t_0} A_{11}(\tau) d\tau + \frac{1}{2!} \left(\int_t^{t_0} A_{11}(\tau) d\tau \right)^2 + \frac{1}{3!} \left(\int_t^{t_0} A_{11}(\tau) d\tau \right)^3 + \dots \right) A_{11}(t) \\ &= -\phi_{11}(t, t_0)A_{11}(t). \end{aligned}$$

Hence, we have

$$\begin{aligned} \phi_{11}(t_0, t)\dot{\phi}_{12}(t, t_0) + \dot{\phi}_{11}(t, t_0)\phi_{12}(t, t_0) &= \phi_{11}(t_0, t)A_{12}(t)\phi_{22}(t, t_0) \\ \frac{d}{dt} (\phi_{11}(t_0, t)\phi_{12}(t, t_0)) &= \phi_{11}(t_0, t)A_{12}(t)\phi_{22}(t, t_0) \\ \int_{t_0}^t d(\phi_{11}(t_0, \tau)\phi_{12}(\tau, t_0)) &= \int_{t_0}^t \phi_{11}(t_0, \tau)A_{12}(\tau)\phi_{22}(\tau, t_0) d\tau \\ \phi_{11}(t_0, t)\phi_{12}(t, t_0) - I &= \int_{t_0}^t \phi_{11}(t_0, \tau)A_{12}(\tau)\phi_{22}(\tau, t_0) d\tau. \end{aligned}$$

Pre-multiply with $\phi_{11}(t, t_0)$ yields

$$\phi_{12}(t, t_0) = \phi_{11}(t, t_0) + \int_{t_0}^t \phi_{11}(t, \tau) A_{12}(\tau) \phi_{22}(\tau, t_0) d\tau.$$

This result is slightly different than the one in the problem. It will be equivalent if we have $\phi_{11}(t, t_0) = 0$. ■

(b) The state transition matrix is

$$\begin{aligned} \phi_{12}(t, t_0) &= \int_{t_0}^t \phi_{11}(t, \tau) A_{12}(\tau) \phi_{22}(\tau, t_0) d\tau \\ &= \int_{t_0}^t 2\tau d\tau = t^2 - t_0^2. \end{aligned}$$

Then for a system with zero initial condition,

$$\begin{aligned} x(t) &= \int_{t_0}^t \begin{bmatrix} 1 & t^2 - \tau^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2\tau \end{bmatrix} d\tau \\ &= \int_{t_0}^t \begin{bmatrix} 2t^2\tau - 2\tau^3 \\ 2\tau \end{bmatrix} d\tau \\ &= \begin{bmatrix} t^2\tau^2 - \frac{1}{2}\tau^4 \\ \tau^2 \end{bmatrix} \Big|_{t_0}^t \\ &= \begin{bmatrix} \frac{1}{2}t^4 - t^2t_0^2 + \frac{1}{2}t_0^4 \\ t^2 - t_0^2 \end{bmatrix}. \end{aligned}$$

6. Building on the theoretical results from Problem 5, find the STM for

$$A(t) = \begin{bmatrix} -2 & 0 & t \\ 2t & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

You should use the result you proved in Problem 5:

$$\phi_{12}(t, t_0) = \int_{t_0}^t \phi_{11}(t, \tau) A_{12}(\tau) \phi_{22}(\tau, t_0) d\tau.$$

Answer:

$$\begin{aligned} \phi_{12}(t, t_0) &= \int_{t_0}^t \phi_{11}(t, \tau) A_{12}(\tau) \phi_{22}(\tau, t_0) d\tau \\ &= \int_{t_0}^t \begin{bmatrix} -2 & 0 & \tau \\ 2\tau & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} d\tau \\ &= \begin{bmatrix} -2\tau & 0 & \frac{1}{2}\tau^2 \\ \tau^2 & -2\tau & \tau \\ 0 & 0 & -2\tau \end{bmatrix} \Big|_{t_0}^t \\ &= \begin{bmatrix} -2(t-t_0) & 0 & \frac{1}{2}(t^2-t_0^2) \\ t^2-t_0^2 & -2(t-t_0) & t-t_0 \\ 0 & 0 & -2(t-t_0) \end{bmatrix}. \end{aligned}$$

7. You are given the following system:

$$\dot{x}(t) = \left(\lambda(t)I + \begin{bmatrix} a(t) & b(t) & c(t) \\ a(t) & b(t) & c(t) \\ a(t) & b(t) & c(t) \end{bmatrix} \bullet \right) x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

where $[M(t)]^\bullet$ denotes the derivative of the matrix, i.e., derivative of the individual entries of the matrix.

(a) Assume that $a(t) + b(t) + c(t) = 0$ for all t . Find a simple expression for $\phi(t, t_0)$.

(b) Given that the control input is $u(t) = 2e^{\lambda t}$, and that $x(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, obtain $x(0)$.

Answer:

(a) Realize that

$$A(t) = \dot{\lambda}(t)I + \begin{bmatrix} a(t) & b(t) & c(t) \\ a(t) & b(t) & c(t) \\ a(t) & b(t) & c(t) \end{bmatrix} = A_1(t) + A_2(t),$$

where $A_1(t)$ and $A_2(t)$ commute. To compute its state transition matrix, we need to evaluate the following exponential matrices

$$e^{\int_{t_0}^t A_1(\tau) d\tau} = e^{\int_{t_0}^t \dot{\lambda}(\tau) d\tau} I = \begin{bmatrix} e^{\lambda(t) - \lambda(t_0)} & 0 & 0 \\ 0 & e^{\lambda(t) - \lambda(t_0)} & 0 \\ 0 & 0 & e^{\lambda(t) - \lambda(t_0)} \end{bmatrix},$$

and

$$e^{\int_{t_0}^t A_2(\tau) d\tau} = I + \int_{t_0}^t A_2(\tau) d\tau + \frac{1}{2!} \left(\int_{t_0}^t A_2(\tau) d\tau \right)^2 + \frac{1}{3!} \left(\int_{t_0}^t A_2(\tau) d\tau \right)^3 + \dots$$

For the latter, first we need to compute

$$\int_{t_0}^t A_2(\tau) d\tau = \begin{bmatrix} a(t) - a(t_0) & b(t) - b(t_0) & c(t) - c(t_0) \\ a(t) - a(t_0) & b(t) - b(t_0) & c(t) - c(t_0) \\ a(t) - a(t_0) & b(t) - b(t_0) & c(t) - c(t_0) \end{bmatrix}.$$

Since $a(t) + b(t) + c(t) = 0$, then

$$\int_{t_0}^t A_2(\tau) d\tau = \begin{bmatrix} a(t) - a(t_0) & b(t) - b(t_0) & -a(t) - b(t) + a(t_0) + b(t_0) \\ a(t) - a(t_0) & b(t) - b(t_0) & -a(t) - b(t) + a(t_0) + b(t_0) \\ a(t) - a(t_0) & b(t) - b(t_0) & -a(t) - b(t) + a(t_0) + b(t_0) \end{bmatrix}.$$

We know that the above matrix is nilpotent for $k = 2$, thus

$$\begin{aligned} e^{\int_{t_0}^t A_2(\tau) d\tau} &= I + \int_{t_0}^t A_2(\tau) d\tau \\ &= \begin{bmatrix} 1 + a(t) - a(t_0) & b(t) - b(t_0) & -a(t) - b(t) + a(t_0) + b(t_0) \\ a(t) - a(t_0) & 1 + b(t) - b(t_0) & -a(t) - b(t) + a(t_0) + b(t_0) \\ a(t) - a(t_0) & b(t) - b(t_0) & 1 - a(t) - b(t) + a(t_0) + b(t_0) \end{bmatrix}. \end{aligned}$$

The state transition matrix can now be formulated as

$$\begin{aligned} \Phi(t, t_0) &= e^{\int_{t_0}^t A_1(\tau) d\tau} e^{\int_{t_0}^t A_2(\tau) d\tau} \\ &= e^{\lambda(t) - \lambda(t_0)} \begin{bmatrix} 1 + a(t) - a(t_0) & b(t) - b(t_0) & -a(t) - b(t) + a(t_0) + b(t_0) \\ a(t) - a(t_0) & 1 + b(t) - b(t_0) & -a(t) - b(t) + a(t_0) + b(t_0) \\ a(t) - a(t_0) & b(t) - b(t_0) & 1 - a(t) - b(t) + a(t_0) + b(t_0) \end{bmatrix}. \end{aligned}$$

(b) The solution of the above system is computed as follows

$$\begin{aligned}
x(t) &= \Phi(t, t_0)x(0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau) d\tau \\
&= e^{\lambda(t)-\lambda(0)} \begin{bmatrix} 1 + a(t) - a(0) & b(t) - b(0) & -a(t) - b(t) + a(0) + b(0) \\ a(t) - a(0) & 1 + b(t) - b(0) & -a(t) - b(t) + a(0) + b(0) \\ a(t) - a(0) & b(t) - b(0) & 1 - a(t) - b(t) + a(0) + b(0) \end{bmatrix} x(0) \\
&\quad + \int_0^t e^{\lambda(t)-\lambda(\tau)} \begin{bmatrix} 1 + a(t) - a(\tau) & b(t) - b(\tau) & -a(t) - b(t) + a(\tau) + b(\tau) \\ a(t) - a(\tau) & 1 + b(t) - b(\tau) & -a(t) - b(t) + a(\tau) + b(\tau) \\ a(t) - a(\tau) & b(t) - b(\tau) & 1 - a(t) - b(t) + a(\tau) + b(\tau) \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (2e^{\lambda(\tau)}) d\tau \\
&= e^{\lambda(t)-\lambda(0)} \begin{bmatrix} 1 + a(t) - a(0) & b(t) - b(0) & -a(t) - b(t) + a(0) + b(0) \\ a(t) - a(0) & 1 + b(t) - b(0) & -a(t) - b(t) + a(0) + b(0) \\ a(t) - a(0) & b(t) - b(0) & 1 - a(t) - b(t) + a(0) + b(0) \end{bmatrix} x(0) \\
&\quad + \int_0^t \begin{bmatrix} 2e^{\lambda(t)} \\ 2e^{\lambda(t)} \\ 2e^{\lambda(t)} \end{bmatrix} d\tau \\
&= e^{\lambda(t)-\lambda(0)} \begin{bmatrix} 1 + a(t) - a(0) & b(t) - b(0) & -a(t) - b(t) + a(0) + b(0) \\ a(t) - a(0) & 1 + b(t) - b(0) & -a(t) - b(t) + a(0) + b(0) \\ a(t) - a(0) & b(t) - b(0) & 1 - a(t) - b(t) + a(0) + b(0) \end{bmatrix} x(0) \\
&\quad + \begin{bmatrix} 2\tau e^{\lambda(t)} \\ 2\tau e^{\lambda(t)} \\ 2\tau e^{\lambda(t)} \end{bmatrix} \Big|_0^t \\
&= e^{\lambda(t)-\lambda(0)} \begin{bmatrix} 1 + a(t) - a(0) & b(t) - b(0) & -a(t) - b(t) + a(0) + b(0) \\ a(t) - a(0) & 1 + b(t) - b(0) & -a(t) - b(t) + a(0) + b(0) \\ a(t) - a(0) & b(t) - b(0) & 1 - a(t) - b(t) + a(0) + b(0) \end{bmatrix} x(0) \\
&\quad + \begin{bmatrix} 2te^{\lambda(t)} \\ 2te^{\lambda(t)} \\ 2te^{\lambda(t)} \end{bmatrix}.
\end{aligned}$$

Substituting $t = 2$ and $x(2) = [1 \ 1 \ 1]^\top$ yields

$$\begin{aligned}
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= e^{\lambda(2)-\lambda(0)} \begin{bmatrix} 1 + a(2) - a(0) & b(2) - b(0) & -a(2) - b(2) + a(0) + b(0) \\ a(2) - a(0) & 1 + b(2) - b(0) & -a(2) - b(2) + a(0) + b(0) \\ a(2) - a(0) & b(2) - b(0) & 1 - a(2) - b(2) + a(0) + b(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \begin{bmatrix} 4e^{\lambda(2)} \\ 4e^{\lambda(2)} \\ 4e^{\lambda(2)} \end{bmatrix} \\
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= e^{\lambda(2)-\lambda(0)} \begin{bmatrix} (1 + a(2) - a(0))x_1(0) + (b(2) - b(0))x_2(0) + (-a(2) - b(2) + a(0) + b(0))x_3(0) + 4e^{\lambda(0)} \\ (a(2) - a(0))x_1(0) + (1 + b(2) - b(0))x_2(0) + (-a(2) - b(2) + a(0) + b(0))x_3(0) + 4e^{\lambda(0)} \\ (a(2) - a(0))x_1(0) + (b(2) - b(0))x_2(0) + (1 - a(2) - b(2) + a(0) + b(0))x_3(0) + 4e^{\lambda(0)} \end{bmatrix}
\end{aligned}$$

From the above equation, one can obtain $x(0)$ where

$$x_1(0) = x_2(0) = x_3(0) = \frac{1 - 4e^{\lambda(0)}}{e^{\lambda(2)-\lambda(0)}},$$

such that

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} \frac{1-4e^{\lambda(0)}}{e^{\lambda(2)-\lambda(0)}} \\ \frac{1-4e^{\lambda(0)}}{e^{\lambda(2)-\lambda(0)}} \\ \frac{1-4e^{\lambda(0)}}{e^{\lambda(2)-\lambda(0)}} \end{bmatrix}.$$

8. Using MATLAB, generate a random LTI dynamical system of 10 states, 5 control inputs, and 3 outputs. Simulate the system given that the inputs are $\text{square}(t)$, $\sin(t)$, $\cos(t)$ and the initial conditions for the system are identically zero. First, simulate the system using the ode45 solver. Then, apply the two discretization methods we discussed in class with variable sampling periods. For example trying sampling periods of $T = 0.01, 0.1$, and 5 seconds. Discuss the outcome that you get between the accurate ODE solver and two discretization methods. Is the discretization always accurate? When does it fail (if it does)? Include your code, plots, and a thorough analysis of the results.

Answer:

Here, a randomly-generated stable system is used. The input is a combination of sine, cosine, and square waves. The output of the LTI system is depicted in Fig. 1. The first DTI is obtained by applying the forward Euler discretization method. Here, several sampling periods are studied. The output of this DTI system is given in Fig. 2. The second DTI is derived using the exact discretization method. Its output is given in Fig. 3.

From these figures, it is apparent that for all DTI systems, the ones with smallest sampling period, which is $T_s = 0.01$ s, have outputs that are similar to that of the LTI system. As the sampling period increases, the DTI systems fail to give the same outputs as the LTI system. This happens because the approximation of the LTI system in discrete-time is worse. At $T_s = 0.5$ s, the first DTI system completely fails, hence it gives unstable output trajectory, whereas the second DTI system still gives stable output trajectory, even if the output is much different compared to the LTI system. This shows the superiority of the exact discretization method over the simpler Euler discretization method. The MATLAB code is given below.

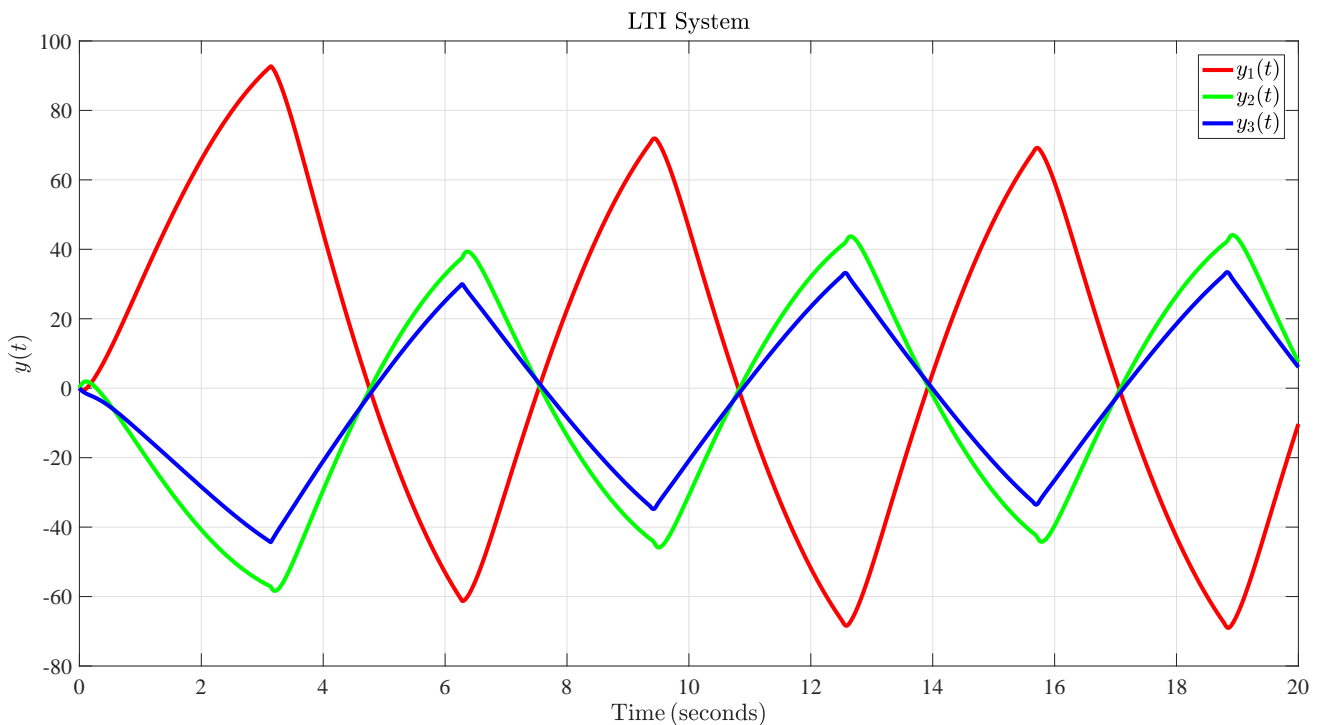


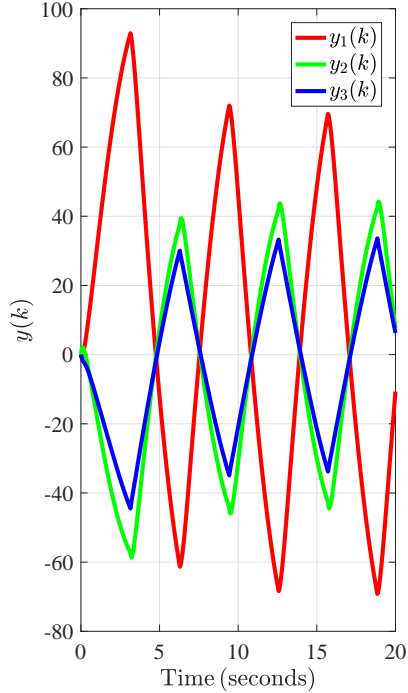
Figure 1: Output trajectory of the LTI system.

```
%% System Initialization
clear all
close all

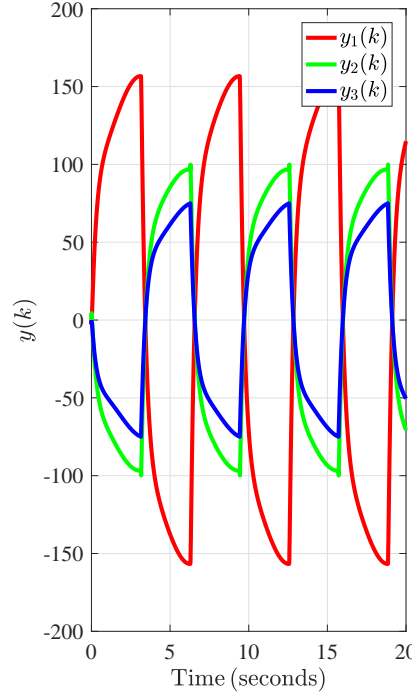
%Figure index
```

DTI System with Euler Discretization Method

$T_s = 0.01$ s



$T_s = 0.1$ s



$T_s = 0.5$ s

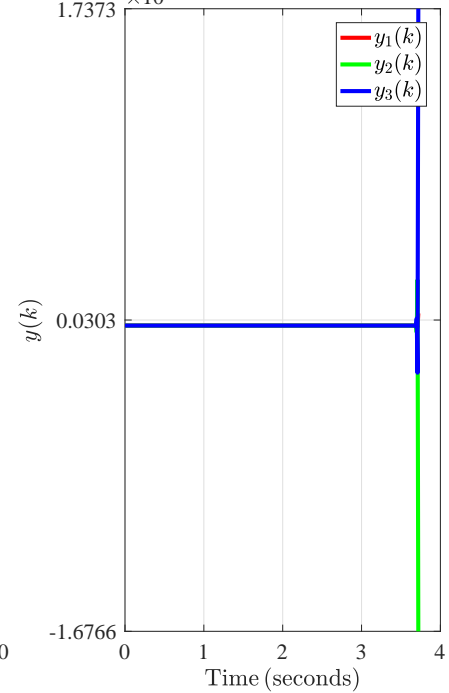


Figure 2: Output trajectory of the DTI system discretized by forward Euler method.

```

figIdx = 1;

%Size
sys.n = 10;
sys.m = 5;
sys.p = 3;

%Random number interval
l = -4;
u = 4;

%Create random matrix
sys.A = 1 + (u-l)*rand(sys.n,sys.n);
sys.B = 1 + (u-l)*rand(sys.n,sys.m);
sys.C = 1 + (u-l)*rand(sys.p,sys.n);

%Get a stable A
for i = 1:sys.n
    sys.A(i,i) = -15 + (-1-(-15))*rand(1,1);
end

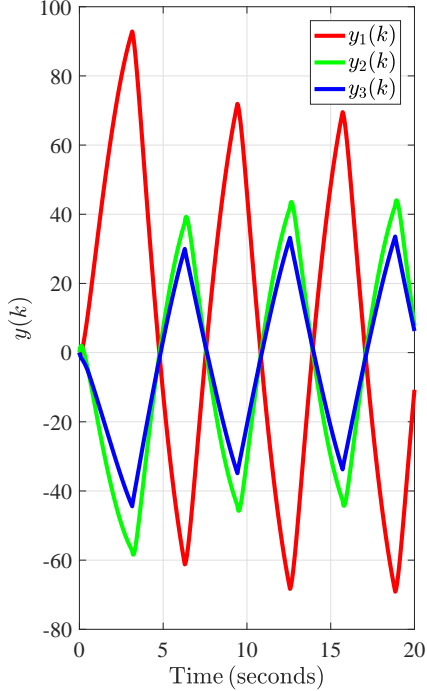
%Save system
save('system_data.mat','sys')

%Initial condition
x0 = zeros(sys.n,1);

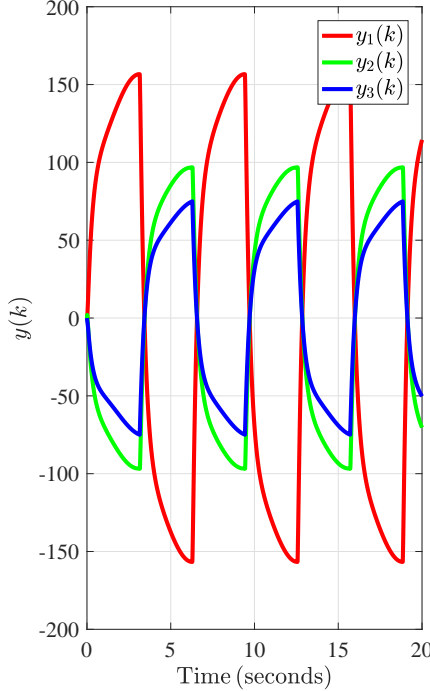
% LTI System
    
```

DTI System with Exact Discretization Method

$T_s = 0.01$ s



$T_s = 0.1$ s



$T_s = 0.5$ s

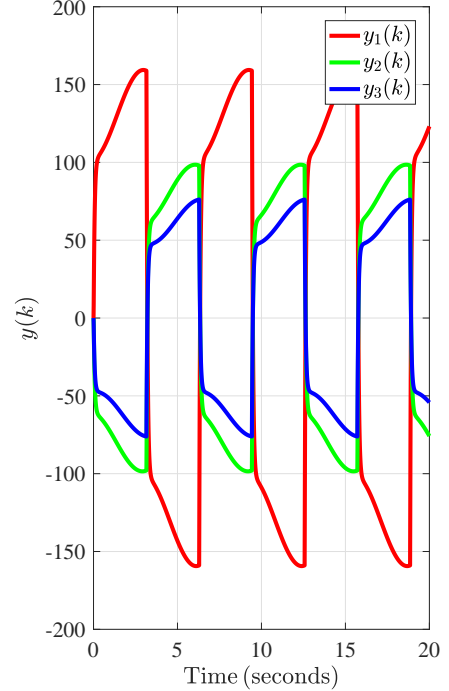


Figure 3: Output trajectory of the DTI system discretized by exact discretization method.

```
%Interval
tff = 20;
period = 0.01;
tspan = 0:period:tff;
```

```
%ODE
[t, x] = ode45(@model, tspan, x0, [], sys);
```

```
%Generate y
y = zeros(length(t),sys.p);
for i=1:length(t)
    xt = x(i,:)';
    y(i,:) = sys.C*xt;
end
```

```
figure(figIdx);
figIdx = figIdx + 1;
fs = 16;
set(gcf,'numbertitle','off','name','Norm')
plot(t,y(:,1),'r',t,y(:,2),'g',t,y(:,3),'b',...
      'LineWidth',3);
title('LTI System','Interpreter','latex')
grid on
xlabel('\textrm{Time\,(seconds)}','interpreter','latex','FontName','Times New Roman','FontSize',fs);
ylabel('$y(t)$','interpreter','latex','FontName','Times New Roman','FontSize',fs);
set(gca,'FontName','Times New Roman','fontSize',fs);
h = legend('$y_1(t)$','$y_2(t)$','$y_3(t)$','Location','northeast');
set(h,'Interpreter','latex')
set(h,'FontSize',fs)
```

```

set(gcf,'color','w');

% DTI Based On Forward Euler Discretization Method, Ts = 0.01 s

%Time sampling
Ts = 0.01;

%Discretize
Ad = eye(sys.n)+sys.A*Ts;
Bd = sys.B*Ts;
Cd = sys.C;

%Interval
tff = 20;
period = 0.01;
tspan = 0:period:tff;

%Control input
u = [];
for i=1:length(tspan)
    ut = [sin(tspan(:,i)); cos(tspan(:,i)); square(tspan(:,i)); cos(tspan(:,i)); sin(tspan(:,i))];
    u = [u; ut'];
end

%start simulation
x = [];
y = [];
xk = x0;
for i=1:length(tspan)
    xkk = Ad*xk + Bd*u(i,:);
    yk = Cd*xk;
    x = [x; xkk'];
    y = [y; yk'];
    xk = xkk;
end

figure(figIdx);
%figIdx = figIdx+1;
fs = 16;
set(gcf,'numbertitle','off','name','Norm')
subplot(1,3,1);
plot(tspan,y(:,1),'r',tspan,y(:,2),'g',tspan,y(:,3),'b',...
    'LineWidth',3);
title({'DTI System with Euler Discretization Method','Ts = 0.01 s'},'Interpreter','latex')
grid on
xlabel('$\text{Time}\,(seconds)$','interpreter','latex','FontName','Times New Roman','FontSize',fs);
ylabel('$y(k)$','interpreter','latex','FontName','Times New Roman','FontSize',fs);
set(gca,'FontName','Times New Roman','fontsize',fs);
h = legend('$y_1(k)$','$y_2(k)$','$y_3(k)$','Location','northeast');
set(h,'Interpreter','latex')
set(h,'FontSize',fs)
set(gcf,'color','w');

% DTI Based On Forward Euler Discretization Method, Ts = 0.1 s

%Time sampling

```

```

Ts = 0.1;

%Discretize
Ad = eye(sys.n)+sys.A*Ts;
Bd = sys.B*Ts;
Cd = sys.C;

%Interval
tff = 20;
period = 0.01;
tspan = 0:period:tff;

%Control input
u = [];
for i=1:length(tspan)
    ut = [sin(tspan(:,i)); cos(tspan(:,i)); square(tspan(:,i)); cos(tspan(:,i)); sin(tspan(:,i))];
    u = [u; ut'];
end

end

%start simulation
x = [];
y = [];
xk = x0;
for i=1:length(tspan)
    xkk = Ad*xk + Bd*u(i,:);
    yk = Cd*xk;
    x = [x; xkk'];
    y = [y; yk'];
    xk = xkk;
end

end

figure(figIdx);
%figIdx = figIdx + 1;
fs = 16;
set(gcf,'numbertitle','off','name','Norm')
subplot(1,3,2);
plot(tspan,y(:,1),'r',tspan,y(:,2),'g',tspan,y(:,3),'b',...
    'LineWidth',3);
title({'Ts = 0.1 s'},'Interpreter','latex')
grid on
xlabel({'$\text{Time}\,(seconds)$'},'interpreter','latex','FontName','Times New Roman','FontSize',fs);
ylabel({'$y(k)$'},'interpreter','latex','FontName','Times New Roman','FontSize',fs);
set(gca,'FontName','Times New Roman','fontsize',fs);
h = legend({'$y_1(k)$','$y_2(k)$','$y_3(k)$'},'Location','northeast');
set(h,'Interpreter','latex')
set(h,'FontSize',fs)
set(gcf,'color','w');

% DTI Based On Forward Euler Discretization Method, Ts = 0.5 s

%Time sampling
Ts = 0.5;

%Discretize
Ad = eye(sys.n)+sys.A*Ts;

```

```

Bd = sys.B*Ts;
Cd = sys.C;

%Interval
tff = 20;
period = 0.01;
tspan = 0:period:tff;

%Control input
u = [];
for i=1:length(tspan)
    ut = [sin(tspan(:,i)); cos(tspan(:,i)); square(tspan(:,i)); cos(tspan(:,i)); sin(tspan(:,i))];
    u = [u; ut'];
end

%start simulation
x = [];
y = [];
xk = x0;
for i=1:length(tspan)
    xkk = Ad*xk + Bd*u(i,:);
    yk = Cd*xk;
    x = [x; xkk'];
    y = [y; yk'];
    xk = xkk;
end

figure(figIdx);
figIdx = figIdx + 1;
fs = 16;
set(gcf,'numbertitle','off','name','Norm')
subplot(1,3,3);
plot(tspan,y(:,1),'r',tspan,y(:,2),'g',tspan,y(:,3),'b',...
    'LineWidth',3);
title({'Ts = 0.5 s'},'Interpreter','latex')
grid on
xlabel({'\textrm{Time\,(seconds)}$'},'interpreter','latex','FontName','Times New Roman','FontSize',fs);
ylabel({'$y(k)$'},'interpreter','latex','FontName','Times New Roman','FontSize',fs);
set(gca,'FontName','Times New Roman','fontsize',fs);
h = legend({'$y_1(k)$','$y_2(k)$','$y_3(k)$'},'Location','northeast');
set(h,'Interpreter','latex')
set(h,'FontSize',fs)
set(gcf,'color','w');

%% DTI Based On Exact Discretization Method, Ts = 0.01 s

%Time sampling
Ts = 0.01;

%Discretize
Ad = expm(sys.A*Ts);
Bd = inv(sys.A)*(Ad-eye(sys.n))*sys.B;
Cd = sys.C;

%Interval
tff = 20;

```

```

period = 0.01;
tspan = 0:period:tff;

%Control input
u = [];
for i=1:length(tspan)
    ut = [sin(tspan(:,i)); cos(tspan(:,i)); square(tspan(:,i)); cos(tspan(:,i)); sin(tspan(:,i))];
    u = [u; ut'];
end

%start simulation
x = [];
y = [];
xk = x0;
for i=1:length(tspan)
    xkk = Ad*xk + Bd*u(i,:);
    yk = Cd*xk;
    x = [x; xkk'];
    y = [y; yk'];
    xk = xkk;
end

figure(figIdx);
%figIdx = figIdx + 1;
fs = 16;
set(gcf,'numbertitle','off','name','Norm')
subplot(1,3,1);
plot(tspan,y(:,1),'r',tspan,y(:,2),'g',tspan,y(:,3),'b',...
    'LineWidth',3);
title({'DTI System with Exact Discretization Method','Ts = 0.01 s'},'Interpreter','latex')
grid on
xlabel({'\textrm{Time}\,(seconds)}$','interpreter','latex','FontName','Times New Roman','FontSize',fs),
ylabel({'$y(k)$','interpreter','latex','FontName','Times New Roman','FontSize',fs);
set(gca,'FontName','Times New Roman','fontsize',fs);
h = legend({'$y_1(k)$','$y_2(k)$','$y_3(k)$','Location','northeast');
set(h,'Interpreter','latex')
set(h,'FontSize',fs)
set(gcf,'color','w');

%% DTI Based On Exact Discretization Method, Ts = 0.1 s

%Time sampling
Ts = 0.1;

%Discretize
Ad = expm(sys.A*Ts);
Bd = inv(sys.A)*(Ad-eye(sys.n))*sys.B;
Cd = sys.C;

%Interval
tff = 20;
period = 0.01;
tspan = 0:period:tff;

%Control input
u = [];

```



```

for i=1:length(tspan)
    ut = [sin(tspan(:,i)); cos(tspan(:,i)); square(tspan(:,i)); cos(tspan(:,i)); sin(tspan(:,i))];
    u = [u; ut'];
end

%start simulation
x = [];
y = [];
xk = x0;
for i=1:length(tspan)
    xkk = Ad*xk + Bd*u(i,:);
    yk = Cd*xk;
    x = [x; xkk'];
    y = [y; yk'];
    xk = xkk;
end

figure(figIdx);
%figIdx = figIdx + 1;
fs = 16;
set(gcf,'numbertitle','off','name','Norm')
subplot(1,3,2);
plot(tspan,y(:,1),'r',tspan,y(:,2),'g',tspan,y(:,3),'b',...
    'LineWidth',3);
title({'Ts = 0.1 s'},'Interpreter','latex')
grid on
xlabel({'\textrm{Time\,(seconds)}$'},'interpreter','latex','FontName','Times New Roman','FontSize',fs)
ylabel({'$y(k)$'},'interpreter','latex','FontName','Times New Roman','FontSize',fs);
set(gca,'FontName','Times New Roman','fontsize',fs);
h = legend({'$y_1(k)$','$y_2(k)$','$y_3(k)$'},'Location','northeast');
set(h,'Interpreter','latex')
set(h,'FontSize',fs)
set(gcf,'color','w');

%% DTI Based On Exact Discretization Method, Ts = 0.5 s

%Time sampling
Ts = 0.5;

%Discretize
Ad = expm(sys.A*Ts);
Bd = inv(sys.A)*(Ad-eye(sys.n))*sys.B;
Cd = sys.C;

%Interval
tff = 20;
period = 0.01;
tspan = 0:period:tff;

%Control input
u = [];
for i=1:length(tspan)
    ut = [sin(tspan(:,i)); cos(tspan(:,i)); square(tspan(:,i)); cos(tspan(:,i)); sin(tspan(:,i))];
    u = [u; ut'];
end

```

```

%start simulation
x = [];
y = [];
xk = x0;
for i=1:length(tspan)
    xkk = Ad*xk + Bd*u(i,:);
    yk = Cd*xk;
    x = [x; xkk'];
    y = [y; yk'];
    xk = xkk;
end

figure(figIdx);
figIdx = figIdx + 1;
fs = 16;
set(gcf,'numbertitle','off','name','Norm')
subplot(1,3,3);
plot(tspan,y(:,1),'r',tspan,y(:,2),'g',tspan,y(:,3),'b',...
    'LineWidth',3);
title({'Ts = 0.5 s'},'Interpreter','latex')
grid on
xlabel('$\text{Time}\,(seconds)$','interpreter','latex','FontName','Times New Roman','FontSize',fs);
ylabel('$y(k)$','interpreter','latex','FontName','Times New Roman','FontSize',fs);
set(gca,'FontName','Times New Roman','fontsize',fs);
h = legend('$y_1(k)$','$y_2(k)$','$y_3(k)$','Location','northeast');
set(h,'Interpreter','latex')
set(h,'FontSize',fs)
set(gcf,'color','w');

%% LTI System Model

%System model
function xdot = model(t, x, sys)

    % Control input
    u = [sin(t); cos(t); square(t); cos(t); sin(t)];

    % System
    xdot = sys.A*x + sys.B*u;

end
%%

```