THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 4
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The objective of this homework is to test your understanding of the content of Module 4. Due date of the homework is: Thursday, October 5th @ 11:59pm.

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1. Consider the following time-varying system:

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{t+1} & 0 \\ -\frac{1}{t+1} & 0 \end{bmatrix} x(t).$$

   (a) Find the state transition matrix.

   (b) Now assume that the system starts from an unknown $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, where $x_1(0) = 1$. Also, assume that $x_2(t = 4) = -1$. Given this, find $x_2(0)$.

   (c) After obtaining $x_2(0)$ and given $x_1(0) = 1$, find a general from of $x(t)$ starting $x(0)$ as initial conditions.

   (d) What happens to $x_1(t)$ and $x_2(t)$ as $t \to \infty$?

2. The STM of $\dot{x}(t) = A(t)x(t)$ is given as follows:

$$\phi(t, t_0) = e^{t_0^2 - t^2} \begin{bmatrix} 1 & \ln\left(\frac{t+1}{t_0+1}\right) \\ 0 & 1 \end{bmatrix}.$$

   (a) Given any STM, how can you obtain $A(t)$ back? Prove that $A(t) = \dot{\phi}(t, t)$ for any LTV system.

   (b) For the STM given in this problem, obtain $A(t)$.

   (c) Find $\phi^{-1}(t, t_0)$

3. The following system

$$\dot{x}(t) = A(t)x(t) + e^{-\alpha^2(t)} \begin{bmatrix} \pi \\ 0 \end{bmatrix} u(t).$$

   The STM for this system is given by:

$$\phi(t, t_0) = e^{\alpha^2(t_0) - \alpha^2(t)} \begin{bmatrix} 1 & 0 \\ \cos(\pi t) - \cos(\pi t_0) & 1 \end{bmatrix}.$$

   (a) Compute $A(t)$.

   (b) Determine the inverse of the STM.

   (c) Find $x(t)$ if $x(t_0) = 0$ and $u(t) = 1$.

4. Find the STM associated with

$$A(t) = \begin{bmatrix} \sin(t) & \cos(t) & \beta \\ 0 & \sin(t) & \cos(t) \\ 0 & 0 & \sin(t) \end{bmatrix}.$$

5. Consider the following dynamical system:

$$\dot{x}(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} x(t).$$

(a) Derive this structure for the STM:

$$\phi(t,t_0) = \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ 0 & \phi_{22}(t,t_0) \end{bmatrix}.$$

You should basically show that $\dot{\phi}_{ii}(t,t_0) = A_{ii}(t)\phi_{ii}(t,t_0)$, and then show a explicit form for $\phi_{12}(t,t_0)$.
Hint: Remember that $\dot{phi}(t,t_0) = A(t)\phi(t,t_0)$ and $\phi(t_0,t_0) = I$. You should use that to prove the following result:

$$\phi_{12}(t,t_0) = \int_{t_0}^{t} \phi_{11}(t,\tau)A_{12}(\tau)\phi_{22}(\tau,t_0)d\tau.$$

(b) Assume that the dynamics for a system are given by:

$$\dot{x}(t) = \begin{bmatrix} 0 & 2t \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2t \end{bmatrix} u(t).$$

Use the results you developed in part (a) to determine $x(t)$ giving that the initial conditions for the system are zero, and $u(t) = 1$, without evaluating this integral $\int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau$.

6. Building on the theoretical results from Problem 5, find the STM for

$$A(t) = \begin{bmatrix} -2 & 0 & t \\ 2t & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

You should use the result you proved in Problem 5:

$$\phi_{12}(t,t_0) = \int_{t_0}^{t} \phi_{11}(t,\tau)A_{12}(\tau)\phi_{22}(\tau,t_0)d\tau.$$

7. You are given the following system:

$$\dot{x}(t) = \left(\dot{\lambda}(t)I + \begin{bmatrix} a(t) & b(t) & c(t) \\ a(t) & b(t) & c(t) \\ a(t) & b(t) & c(t) \end{bmatrix}^{\bullet} \right) x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

where $\left[M(t)\right]^{\bullet}$ denotes the derivative of the matrix, i.e., derivative of the individual entries of the matrix.

(a) Assume that $a(t) + b(t) + c(t) = 0$ for all t. Find a simple expression for $\phi(t,t_0)$.

(b) Given that the control input is $u(t) = 2e^{\lambda(t)}$, and that $x(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, obtain $x(0)$.

8. Using MATLAB, generate a random LTI dynamical system of 10 states, 5 control inputs, and 3 outputs. Simulate the system given that the inputs are square(t), $\sin(t)$, $\cos(t)$ and the initial conditions for the system are identically zero. First, simulate the system using the ode45 solver. Then, apply the two discretization methods we discussed in class with variable sampling periods. For example trying sampling periods of $T = 0.01, 0.1$, and 5 seconds. Discuss the outcome that you get between the accurate ODE solver and two discretization methods. Is the discretization always accurate? When does it fail (if it does)? Include your code, plots, and a thorough analysis of the results.