

Due date of the homework is: Wednesday, November 8th @ 11:59pm.

1. Prove that the system represented in the controllable canonical form is always controllable.

Answer: The transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n}$$

has the controllable canonical form with matrices A and B as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \dots & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Forming the controllability matrix, we have

$$C_o = [B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad \dots \quad A^{n-1}B]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & -a_1 \\ 0 & 0 & 0 & 0 & 0 & \dots & a_1^2 - a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & -a_1 & a_1^2 - a_2 & \dots & * \\ 0 & 1 & -a_1 & a_1^2 - a_2 & * & \dots & * \\ 1 & -a_1 & a_1^2 - a_2 & * & * & \dots & * \end{bmatrix}.$$

where '*' indicates the non-zero value. Since the controllability matrix of this form will always have the dimension of $n \times n$, thanks to the unique structure of C_o where ones appear on the anti-diagonal elements, then $\text{Rank}(C_o) = n$, which finally implies that the controllable canonical form is always controllable. ■

2. Show that the controller design

$$u(t) = -B^\top e^{A^\top(t_f-t)} W^{-1}(t_f) [e^{At_f} x_0 - x_{t_f}]$$

steers the system from $x(t_0) = x_0$ to $x(t_f) = x_{t_f}$.

Answer: The solution of any CT-LTI with nonzero input is

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau.$$

Suppose the system starts from $x(0) = x_0$ and we want to achieve x_{t_f} at $t = t_f$. Hence, we want to prove that $x(t_f) = x_{t_f}$. With the aforementioned control input, then

$$\begin{aligned} x(t_f) &= e^{At_f} x_0 + \int_0^{t_f} e^{A(t_f-\tau)} B u(\tau) d\tau \\ &= e^{At_f} x_0 + \int_0^{t_f} e^{A(t_f-\tau)} B (-B^\top e^{A^\top(t_f-\tau)} W^{-1}(t_f) [e^{At_f} x_0 - x_{t_f}]) d\tau \\ &= e^{At_f} x_0 - \int_0^{t_f} e^{A(t_f-\tau)} B B^\top e^{A^\top(t_f-\tau)} d\tau (W^{-1}(t_f) [e^{At_f} x_0 - x_{t_f}]) \end{aligned}$$

$$\begin{aligned}
&= e^{At_f} x_0 - W(t_f)(W^{-1}(t_f) [e^{At_f} x_0 - x_{t_f}]) \\
&= e^{At_f} x_0 - [e^{At_f} x_0 - x_{t_f}] \\
&= e^{At_f} x_0 - e^{At_f} x_0 + x_{t_f} \\
&= x_{t_f}.
\end{aligned}$$

■

3. You are given the following CT LTI system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u(t).$$

We wish to find a state feedback controller $u = Kx$ (not $u = -Kx$) such that $A_{cl} = A + BK$ is **block diagonal** with eigenvalues $\lambda_{1,2} = \{2, 3\}$ assigned to the first diagonal block, and eigenvalues $\lambda_{3,4} = \{0, 1\}$ assigned to second diagonal block. Note that your K matrix can be written as:

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}.$$

Answer: A_{cl} is equal to

$$A_{cl} = A + BK = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 + 1 & k_3 + 1 & k_4 + 1 \\ 1 & 0 & 0 & 0 \\ k_5 + 1 & k_6 + 1 & k_7 & k_8 \\ k_5 + 1 & k_6 + 1 & k_7 + 1 & k_8 \end{bmatrix}.$$

The first diagonal matrix is

$$A_{cl1} = \begin{bmatrix} k_1 & k_2 + 1 \\ 1 & 0 \end{bmatrix},$$

where

$$\text{Det}(A_{cl1} - \lambda I) = \lambda^2 - k_1\lambda - (k_2 + 1),$$

such that it is equal to

$$(\lambda - 2)(\lambda - 3) = \lambda^2 - 5\lambda + 6 = \lambda^2 - k_1\lambda - (k_2 + 1) = 0, \quad \Rightarrow \quad k_1 = 5, k_2 = -7.$$

The second diagonal matrix is

$$A_{cl2} = \begin{bmatrix} k_7 & k_8 \\ k_7 + 1 & k_8 \end{bmatrix},$$

where

$$\text{Det}(A_{cl2} - \lambda I) = \lambda^2 - (k_7 + k_8)\lambda - k_8,$$

such that it is equal to

$$(\lambda)(\lambda - 1) = \lambda^2 - \lambda = \lambda^2 - (k_7 + k_8)\lambda - k_8 = 0, \quad \Rightarrow \quad k_7 = 1, k_8 = 0.$$

With the remaining terms $k_3 = k_4 = k_5 = k_6 = -1$. Hence, the control law is

$$u(t) = \begin{bmatrix} 5 & -7 & -1 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} x(t).$$

4. Answer the following questions for this system:

$$x(k+1) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k).$$

- (a) Is the system controllable? Show this result via the first three controllability tests. You can use MATLAB in this problem to find the eigenvalues and the eigenvectors.
- (b) Is the system stabilizable? If so, design a state feedback controller $u(k) = -Kx(k)$ that would shift the unstable eigenvalues to stable locations which are $\lambda_{cl}(A) = \{-1, -0.5, 0.5\}$. Can you obtain such a state feedback controller? Here $K \in \mathbb{R}^{1 \times 3}$.
- (c) Consider that $x(0) = 0$. Obtain the reachable subspace \mathcal{R}_k of the system at $k = 1, 2, 3, \dots$. Recall that the reachable subspace is

$$\mathcal{R}_k = \text{Range-Space}([B \quad AB \quad A^2B \quad \dots \quad A^{k-1}B]).$$

- (d) Can you find a control sequence $(u(0), u(1), \dots, u(n-1))$ that can drive the system from $x(0) = 0$ to $x(n) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in the least possible time-steps n . You can start by trying $n = 1$ then $n = 2$, etc...

Answer:

- (a) First, by the controllability matrix, we have

$$Co = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}.$$

Since $\text{Rank}(Co) = 3$, then the system is controllable.

Next, the eigenvalues of A are $\Lambda = \{-1, 1, 2\}$. By using the PBH test, we get

- $\text{Rank}([\lambda_1 I - A \quad B]) = \text{Rank} \left(\begin{bmatrix} 2 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \right) = 3$
- $\text{Rank}([\lambda_2 I - A \quad B]) = \text{Rank} \left(\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \right) = 3$
- $\text{Rank}([\lambda_3 I - A \quad B]) = \text{Rank} \left(\begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = 3$

Since all the tests give full rank, then the system is controllable.

Last, the left eigenvectors of A are

$$W = \begin{bmatrix} 0.4082 & 0.7071 & 0 \\ 0.8165 & 0 & 0 \\ -0.4082 & -0.7071 & 1 \end{bmatrix},$$

such that, multiplying them with B yields

- $W(1)^\top B = [0.4082 \quad 0.8165 \quad -0.4082] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -0.4082$
- $W(2)^\top B = [0.7071 \quad 0 \quad -0.7071] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -0.7071$
- $W(3)^\top B = [0 \quad 0 \quad 1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1.$

Since all of them are nonzero, then the system is controllable.

- (b) Since the system is controllable, then it is also stabilizable. The closed loop system can be written as

$$A_{cl} = A - BK = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ -k_1 & -k_2 & 2 - k_3 \end{bmatrix}.$$

The characteristic equation of A_{cl} is

$$\lambda^3 + (-k_2 + k_3 - 2)\lambda^2 + (k_1 - 1)\lambda + (k_1 - k_3 + 2) = 0.$$

For the closed-loop to have poles at $\{-1, -0.5, 0.5\}$, then

$$\lambda^3 + (-k_2 + k_3 - 2)\lambda^2 + (k_1 - 1)\lambda + (k_1 - k_3 + 2) = \lambda^3 + \lambda^2 - \frac{1}{4}\lambda - \frac{1}{4} = 0,$$

Which gives

$$K = \left[\frac{3}{4} \quad 0 \quad 3 \right].$$

- (c) First, since the matrix A has size of 3×3 , let's see its reachable subspace for $k = 3$.

$$\text{Range}([B \quad AB \quad A^2B]) = \text{Range} \left(\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \right).$$

By using the reduced column echelon form, we get

$$\text{Range} \left(\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \right) = \text{Range} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad a, b, c \in \mathbb{R}.$$

Since for $k = 3$ the reachable subspace spans \mathbb{R}^3 , then for any $k \geq 3$, the reachable subspace also spans \mathbb{R}^3 .

- (d) Since the reachable subspace spans \mathbb{R}^3 for $k = 3$, then the states can be transferred to anywhere in \mathbb{R}^3 for $k = 3$. Moreover, since $B = [0 \quad 0 \quad 1]^\top$, there is no control input that is able to transfer the states to $[1 \quad 1 \quad 0]^\top$ at $k = 1$. However, at $k = 2$, we have

$$\begin{aligned} x(2) &= Bu(1) + ABu(0) \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(1) + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} u(0). \end{aligned}$$

From the equation above, one can set $u(1) = -2$ and $u(0) = 1$ for the states to reach $[1 \quad 1 \quad 0]^\top$.

5. Show that if we discretize a system $\dot{x}(t) = Ax(t) + Bu(t)$ to $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$, then if the discretized system defined by (\tilde{A}, \tilde{B}) is controllable, then so should the continuous system defined by the pair (A, B) .

Hint: You can prove this result by contradiction and by using the eigenvector test for controllability.

Answer: Suppose the continuous time is uncontrollable, then there exist at least one left eigenvector that corresponds to a particular eigenvalue λ_j such that $w_j^\top B = 0$. Without loss of generality, the forward Euler approximation of the continuous time is given as

$$\tilde{A} = I + AT, \quad \tilde{B} = BT,$$

7. Consider the following system

$$\dot{x}(t) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -3 \end{bmatrix} u(t).$$

- (a) Is the system controllable?
- (b) Find the controllability subspace.
- (c) Is the system stabilizable?
- (d) Is there a state feedback gain such that $A - BK$ has eigenvalues $\{-1, 1\}$?
- (e) Suppose that $x(0) = [1 \ 0]^\top$ and $u(t) = 0$. Find $y(t)$ if $y(t) = Cx(t) = [3 \ 1] x(t)$.

Answer:

- (a) No, the system is uncontrollable because

$$\text{Rank} [B \ AB] = \text{Rank} \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} = 1.$$

- (b) using the reduced row echelon, we get

$$\text{Range} \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} = \text{Range} \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix} = c \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad c \in \mathbb{R}.$$

- (c) No, because according to the PBH test for unstable eigenvalue, since the eigenvalues of A are -1 and 2, we get

$$\text{Rank}([2I - A \ B]) = \text{Rank} \left(\begin{bmatrix} 0 & -1 & 1 \\ 0 & 3 & -3 \end{bmatrix} \right) = 1.$$

- (d) The closed-loop dynamics is

$$A_{cl} = A - BK = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} 2 - k_1 & 1 - k_2 \\ 3k_1 & -1 + 3k_2 \end{bmatrix},$$

whereas its characteristic polynomial is

$$\text{Det}(A_{cl} - \lambda I) = \lambda^2 - (k_1 - 3k_2 - 1)\lambda - (-2k_1 + 6k_2 - 2) = 0.$$

The characteristic polynomial with desired eigenvalues at $\{-1, 1\}$ is $\lambda^2 - 1 = 0$. From these two equations, we get a system of linear equations

$$\begin{aligned} k_1 - 3k_2 &= 1 \\ k_1 - 3k_2 &= 0. \end{aligned}$$

Since the above linear equations are inconsistent, it has no solution. Hence there is no state feedback that can move the closed-loop eigenvalues to $\{-1, 1\}$.

- (e) To answer this problem, we need to compute the matrix exponential for A . Since A is diagonalizable, then

$$e^{At} = T e^{Dt} T^{-1} = \begin{bmatrix} 1 & -0.3162 \\ 0 & 0.9487 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0.3333 \\ 0 & 1.0541 \end{bmatrix} = \begin{bmatrix} e^{2t} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} \\ 0 & e^{-t} \end{bmatrix}.$$

We finally get

$$y(t) = Cx(t) = C e^{At} x(0) = [3 \ 1] \begin{bmatrix} e^{2t} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3e^{2t}.$$

8. Coding problem—horray! Ok, so in this problem I want you to write MATLAB function that takes as an input (potentially large) matrices A and B and checks for the four tests of controllability (T1—T4) we learned in class. You can assume that the system is continuous time. Some instructions:
- Implement the four tests as discussed in class in the most efficient way.
 - The function should return a 4-digit binary string with $\{1,1,1,1\}$ denoting that the system is controllable and the four tests all yielded a '1' meaning the system is controllable or $\{0,0,0,0\}$ denoting the system is not controllable. If your code returns something other than that, then you've probably done something wrong.
 - The function should also return the computational time required to run each of the methods (T1—T4). You should start by trying your code for random small-scale dynamic systems, and then try a rather large-scale system with millions of states and tens of thousands of control inputs. The computational time should be returned in an array with four numbers—each number denoting how many seconds it took to run each test. See MATLAB's help files to compute computational time for any function.
 - Your MATLAB function essentially has four other functions that you call for each test. You can assume that $t_f = 10$ for Test 4 (the Gramian test).
 - I'll give extra credits how try hard and do a good job.
 - Please upload: (a) a PDF for the solutions of the homework, and (b) a single m-file with the code that returns the above outputs. Not two PDFs and four m-files, a single PDF and a single m-file—please. :)

Answer: The MATLAB code for the function is given as follows

```
%Homework 6, Problem 8
%Author: Sebastian A. Nugroho

function [Out,Time] = Hw6_Pr8_Sebastian(A,B)

%Dimension
n = size(A,1);
m = size(B,2);

%Initialization
Out = [0 0 0 0];
Time = [0 0 0 0];

%First method - Controllability matrix
tic;
X = B;
Co = [X];
for i = 1:n-1
    Y = A*X;
    Co = [Co Y];
    X = Y;
end
tol = 10^6;
Co(isnan(Co)) = 0;
Co(~isfinite(Co)) = 10^6;
if rank(Co,tol) == n
    Out(1) = 1;
else
    Out(1) = 0;
end
Time(1) = toc;
```

```

%Second method - PBH test
tic;
eigA = eig(A);
I = eye(n);
isCon = 1;
for i = 1:n
    if rank([eigA(i)*I-A B]) ~= n
        isCon = 0;
        break;
    end
end
Out(2) = isCon;
Time(2) = toc;

%Third method - Left eigenvector
tic;
[V,D,W] = eig(A);
I = eye(n);
isCon = 1;
for i = 1:n
    wt = W(:,i);
    if isempty(wt'*B) == true
        isCon = 0;
        break;
    end
end
Out(3) = isCon;
Time(3) = toc;

%Fourth method - Controllability gramian
tic;
f = @(tau) expm(A*tau)*B*B'*expm(A'*tau);
fx = integral(f,0,10,'ArrayValued',true);
if (det(fx) == 0)
    Out(4) = 0;
else
    Out(4) = 1;
end
Time(4) = toc;

end

For example, for  $n = 500$  and  $m = 50$ , it returns

Out =

    1    1    1    1

Time =

    2.7061    46.1331    0.2170    36.8057

```