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EE 3413
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HW#9

1 $y^3(t) + 6y''(t) + 11y'(t) + 6y(t) = 6u(t)$

$$s^3Y(s) - \cancel{s^2y(0)} - \cancel{sy'(0)} - \cancel{y''(0)} + 6(s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)}) + 11(sY(s) - \cancel{y(0)}) + 6Y(s) = 6u(s)$$

zero initial conditions

$$(s^3 + 6s^2 + 11s + 6)Y(s) = 6u(s)$$

(a)

$$H(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^3+6s^2+11s+6}$$

(b)

$$\frac{\overset{0s^3 + 0s^2 + 0s + 6}{b_0s^n + b_1s^{n-1} + b_2s^{n-2} + b_3s^{n-3}}}{\underset{s^3 + 6s^2 + 11s + 6}{s^n + a_1s^{n-1} + a_2s^{n-2} + a_3s^{n-3}}}$$

$b_0 = 0$ $\quad a_1 = 6 \quad n = 3$
$b_1 = 0$ $\quad a_2 = 11$
$b_2 = 0$ $\quad a_3 = 6$
$b_3 = 6$

Controllable Canonical form:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}6 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

Observable Canonical form:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 0 & -6\\ 1 & 0 & -11\\ 0 & 1 & -6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}6\\ 0\\ 0\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

$s^3+6s^2+11s+6$

Factors of 6: $\pm 1, 2, 3, 6$

$$\cancel{\begin{array}{r|rrrr} 1 & 1 & 6 & 11 & 6 \\ & \downarrow & 1 & 7 & 18 \\ \hline & 1 & 7 & 18 & 24 \end{array}}$$

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & \downarrow & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array} \checkmark$$

$x=-1$ $\quad$ x^2+5x+6

$(x+1)$ $\quad$ $(x+3)(x+2)$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^3+6s^2+11s+6}$$

$$= \frac{6}{(s+1)(s+3)(s+2)}$$

$$6 = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$6 = A(s+3)(s+2) + B(s+1)(s+2) + C(s+1)(s+3)$$

$s=-2$

$6 = A(0) + B(0) + C(-1)$

$-6 = C$

$s=-1$

$6 = A(2)$

$3 = A$

$s=-3$

$6 = A(0) + 2B + C(0)$

$3 = B$

$$6 = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$= \frac{3}{s+1} + \frac{-6}{s+2} + \frac{3}{s+3}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \frac{c_3}{s+p_3}$$

Diagonal Canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\boxed{2}$ $A = \begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $D = 0$

$$\frac{Y(s)}{U(s)} = C\,(sI_n - A)^{-1} B + D$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \Big(\underbrace{s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}}_{\begin{bmatrix} s-1 & -2 \\ 4 & s-3 \end{bmatrix}^{-1}} \Big)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0$$

$$* = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{(s-1)(s-3)+8} \begin{bmatrix} s-3 & 2 \\ -4 & s-1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix} \frac{\begin{bmatrix} s-3 & 2 \\ -4 & s-1 \end{bmatrix}}{s^2-4s+11}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0$$

$$= \underbrace{\frac{\begin{bmatrix} s-7 & s+1 \end{bmatrix}}{s^2-4s+11} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} + 0$$

$$= \frac{3s+5}{s^2-4s+11} \leftarrow \text{transfer function}$$

This should be 3s-5, not 3s+5.

$b_0 = 0$ $\quad a_1 = -4$

$b_1 = 3$ $\quad a_2 = 11$

$b_2 = -5$

Controllable Canonical form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -11 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$\boxed{3}$ $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Eigenvalues:

$\det(A - \lambda I_n) = 0$

$\det\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$

$\det \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = 0$

$* \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$-\lambda(-3-\lambda) - (1 \cdot -2)$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\lambda_{1,2} = -1, -2$

Eigenvectors

$(A - \lambda_n)V_n = 0$

$\lambda_1 = -1$

$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} v_1 = 0$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = -2$

$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} V_2 = 0$

$V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Diagonal form:

$A = TDT^{-1}$

$A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

e^{At}:

$e^{At} = T e^{\lambda t} T^{-1}$

$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2e^{-t} & e^{-t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix}$

$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$

$\boxed{4}$

$$\dot{x}(t) = Ax(t)$$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x(t) = e^{At} x_0$$

$$= \begin{bmatrix} -2e^{-t} + e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-t} + e^{-2t} + 3e^{-t} - 3e^{-2t} \\ 2e^{-t} - 2e^{-2t} - 3e^{-t} + 6e^{-2t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} - 2e^{-2t} \\ -e^{-t} + 4e^{-2t} \end{bmatrix}$$

$\boxed{5}$

$$u(t) = 2 + 5e^{-t}$$

$$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B(u(\tau))\,d\tau$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} u(\tau)$$

$$= \begin{bmatrix} e^{-t} - 2e^{-2t} \\ -e^{-t} + 4e^{-2t} \end{bmatrix} + \int_0^t (2+5e^{-\tau})\begin{bmatrix} 4e^{-(t-\tau)} - 3e^{-2(t-\tau)} \\ -4e^{-(t-\tau)} + 6e^{-2(t-\tau)} \end{bmatrix}$$

$$\int_0^t \begin{bmatrix} 8e^{-(t-\tau)} + 20e^{-t} - 6e^{-2(t-\tau)} - 15e^{-2t+\tau} \\ -8e^{-(t-\tau)} - 20e^{-t} + 12e^{-2(t-\tau)} + 30e^{-2t+\tau} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} e^{-t} - 2e^{-2t} \\ -e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} 8e^{-t+\tau} + 20\tau e^{-t} - 3e^{-2t+2\tau} - 15e^{-2t+\tau} \\ -8e^{-t+\tau} - 20\tau e^{-t} + 6e^{-2t+2\tau} + 30e^{-2t+\tau} \end{bmatrix}\Bigg|_{\tau=0}^{\tau=t}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} - 2e^{-2t} \\ -e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} 20te^{-t} + 18e^{-2t} - 23e^{-t} + 5 \\ -20te^{-t} - 36e^{-2t} + 38e^{-t} - 2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 20te^{-t} + 16e^{-2t} - 22e^{-t} + 5 \\ -20te^{-t} - 32e^{-2t} + 37e^{-t} - 2 \end{bmatrix}$$

Command Window

New to MATLAB? See resources for Getting Started.

```
>> A = [0 1; -2 -3];
>> B = [1;2];
>> D = 0;
>> Xo = [-1;3];
>> syms t x;
>> Q = expm(A*t)*Xo

Q =

 exp(-t) - 2*exp(-2*t)
 4*exp(-2*t) - exp(-t)

>> u(x) = 2 + 5*exp(-x)

u(x) =

5*exp(-x) + 2

>> int(expm(A*(t-x))*B*u(x),x,0,t)

ans =

 18*exp(-2*t) - 23*exp(-t) + 20*t*exp(-t) + 5
 38*exp(-t) - 36*exp(-2*t) - 20*t*exp(-t) - 2

>> X(t) = Q + ans

X(t) =

 16*exp(-2*t) - 22*exp(-t) + 20*t*exp(-t) + 5
 37*exp(-t) - 32*exp(-2*t) - 20*t*exp(-t) - 2

>>
```