Your Name: SOLUTIONS

Your Signature:

- **Exam duration:** 1 hour and 15 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place $\boxed{\text{a box around your final answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 6 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.
- Question 6 is a bonus question. You do not need to answer it.

Question Number	Maximum Points	Your Score
1	20	
2	20	
3	25	
4	20	
5	10	
Total	100	

1. (20 total points) Find the Laplace transform or the inverse Laplace transform for the following functions. You may use the LT table.

   (a) (5 points) $f_1(t) = e^{2t}\cos(5t) + e^{-3t}\sinh(10t)$.

   $$\boxed{F_1(s) = \frac{s-2}{(s-2)^2+5^2} + \frac{10}{(s+3)^2-10^2}.}$$

   (b) (5 points) $F_2(s) = \dfrac{\sqrt{45}s}{(s^2+16)^2}$.

   $$\boxed{f_2(t) = \frac{3}{8}\sqrt{5}\cdot t\sin(4t).}$$

   (c) (5 points) $f_3(t) = e^{2t}(t^3+5t-2)$.

   $$\boxed{F_3(s) = \frac{5}{(s-2)^2} - \frac{2}{s-2} + \frac{6}{(s-2)^4}.}$$

   (d) (5 points) $F_4(s) = \dfrac{s+1}{(s-2)(s+2)}$.

   $$\boxed{f_4(t) = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{2t}.}$$

2. (20 total points) The following ODE is given:

$$y''(t) - y'(t) - 2y(t) = e^{2t}.$$

(a) (20 points) Given that $y(0) = 0$ and $y'(0) = 1$, find the solution $y(t)$ to the above ODE via Laplace transforms.

Taking the Laplace transform for the ODE, and considering the given initial conditions, we obtain:

$$s^2Y(s) - 1 - sY(s) - 2Y(s) = \frac{1}{s-2}$$

or

$$Y(s) = \frac{s-1}{(s-2)(s^2-s-2)} = \frac{s-1}{(s-2)^2(s+1)}.$$

Applying partial fraction expansion, we obtain:

$$Y(s) = \frac{s-1}{(s-2)(s^2-s-2)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2},$$

where $A = -\frac{2}{9}, B = \frac{2}{9}$, and $C = \frac{1}{3}$.

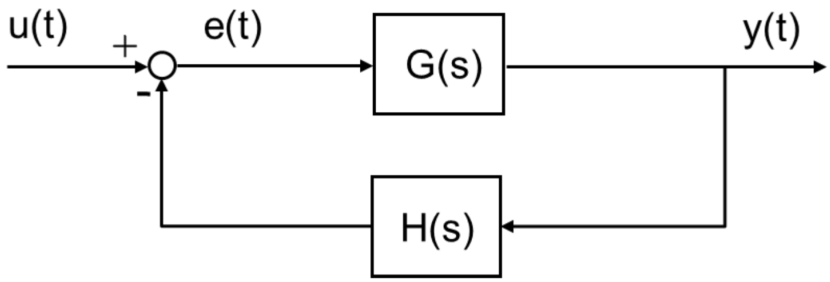
Therefore,

$$\boxed{y(t) = -\frac{2}{9}e^{-t} + \frac{2}{9}e^{2t} + \frac{1}{3}te^{2t}.}$$

3. (25 total points) For the system given in the below figure, assume that:

$$G(s) = \frac{1}{(s-1)(s+3)},$$

$$H(s) = 4.$$



(a) (5 points) Find the transfer function $\frac{Y(s)}{U(s)}$ in the most simplified form.

$$\boxed{\frac{Y(s)}{U(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{1}{(s-1)(s+3)}}{1+4\frac{1}{(s-1)(s+3)}} = \frac{1}{(s-1)(s+3)+4} = \frac{1}{s^2+2s+1}.}$$

(b) (5 points) Find $Y(s)$ if $u(t) = 1$. DO NOT compute $y(t)$.

$$\boxed{Y(s) = \frac{U(s)}{s^2+2s+1} = \frac{1}{s(s^2+2s+1)} = \frac{1}{s(s+1)^2}.}$$

(c) (5 points) What are the poles of $Y(s)$? Does the final value of $y(t)$ exist (i.e., $y(\infty)$)? If it does, find it via the **final value theorem**. Otherwise, tell me why it doesn't.

$\boxed{\text{The poles of } Y(s) \text{ are: } 0, -1, -1. \text{ Therefore, the final value theorem applies:}}$

$$\boxed{y(\infty) = lim_{s\to 0}\, sY(s) = 1.}$$

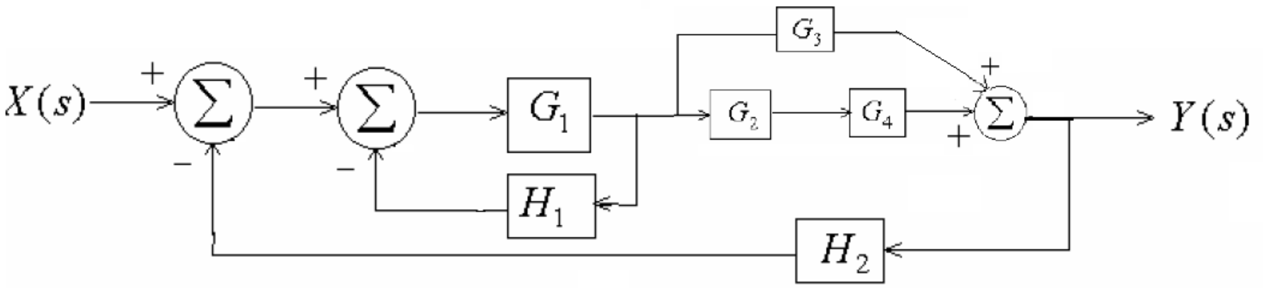
(d) (10 points) Obtain $\frac{E(s)}{U(s)}$, then find $E(s)$ for the given $u(t) = 1$. Does the final value of $e(t)$ exist (i.e., $e(\infty)$)? If it does, find it via the **final value theorem**. Otherwise, tell me why it doesn't.

$$\boxed{\frac{E(s)}{U(s)} = \frac{1}{1+G(s)H(s)} \Rightarrow E(s) = \frac{(s-1)(s+3)}{(s-1)(s+3)+4}\cdot\frac{1}{s} = \frac{(s-1)(s+3)}{s(s+1)^2}.}$$

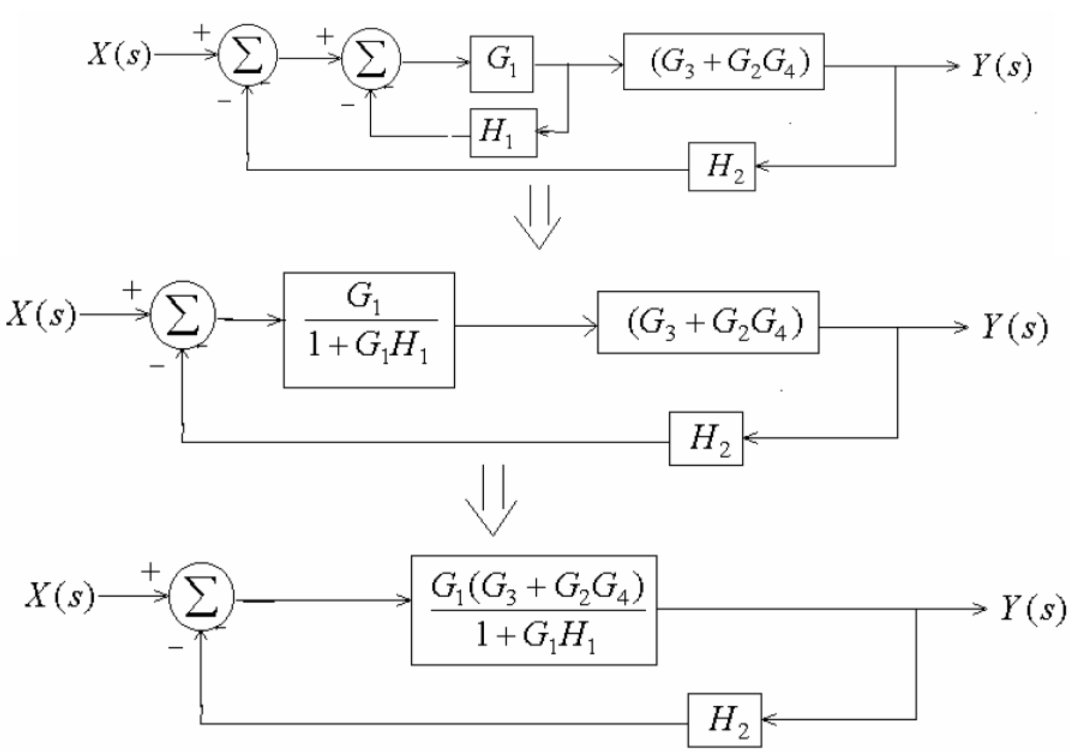
$\boxed{\text{The poles of } E(s): 0, -1, -1. \text{ Therefore, the final value theorem applies:}}$

$$\boxed{e(\infty) = lim_{s\to 0}\, sE(s) = -3.}$$

4. (20 total points) You are given the following block diagram.

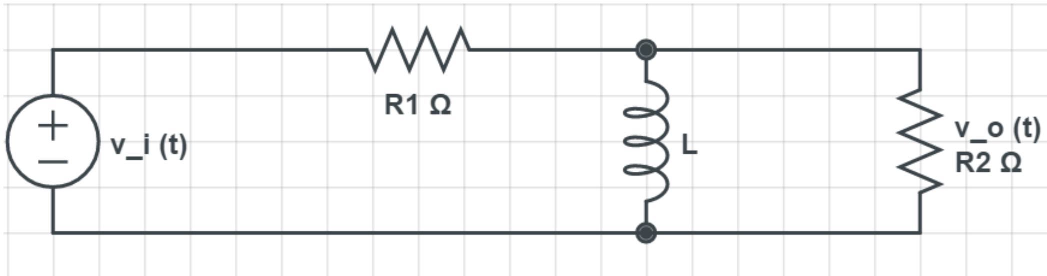


(a) (20 points) Find $\dfrac{Y(s)}{X(s)}$ for the above system. Show your work.



$$\frac{Y(s)}{X(s)} = \frac{\dfrac{G_1(G_3+G_2G_4)}{1+G_1H_1}}{1+\dfrac{G_1(G_3+G_2G_4)}{1+G_1H_1}H_2} = \frac{G_1(G_3+G_2G_4)}{1+G_1H_1+H_2G_1(G_3+G_2G_4)}$$

5. (10) You are given the following RLC circuit.



(a) (10 points) Derive the transfer function $\frac{V_o(s)}{V_i(s)}$ in terms of R_1, L, and R_2. Show your work.

Let I, I_1, I_2 be the currents across R_1, R_2, L, respectively. This is a very similar example to the one from the recitation session. Applying KCL and KVL, we obtain:

$$\boxed{I(s) = I_1(s) + I_2(s),\ V_i(s) = R_1 I(s) + LsI_2(s),\ LsI_2(s) = R_2 I_1(s)}$$

Therefore, $I_1(s) = (Ls/R_2)I_2(s)$, and hence:

$$I(s) = I_1(s) + I + I_2(s) = (Ls/R_2)I_2(s) + I_2(s) \Rightarrow I_2(s) = \frac{1}{(Ls/R_2)+1} I(s).$$

Thus:

$$V_i(s) = R_1 I(s) + LsI_2(s) \Rightarrow V_i(s) = \left(R_1 + \frac{Ls}{(Ls/R_2)+1}\right) I(s).$$

We also know that

$$V_o(s) = R_2 I_1(s) = LsI_2(s) = \frac{Ls}{(Ls/R_2)+1} I(s).$$

Dividing the two equations above, we obtain:

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{Ls}{(Ls/R_2)+1} I(s)}{\left(R_1 + \frac{Ls}{(Ls/R_2)+1}\right) I(s)} = \frac{\frac{Ls}{(Ls/R_2)+1}}{\left(R_1 + \frac{Ls}{(Ls/R_2)+1}\right)}$$

$$= \frac{Ls}{\left(L + \frac{R_1 L}{R_2}\right) s + R_1} = \frac{R_2 Ls}{(R_1+R_2)Ls + R_1 R_2}.$$

6. (15) ***[Bonus Question: Do not answer this before finishing the first five exam questions.]***
   (a) (15 points) Prove the initial value theorem:

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s).$$

$\boxed{\text{Check the uploaded handout on Blackboard.}}$