Module 02
Laplace Transforms, Transfer Functions & ODEs

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February 3, 2016

Module 02 Outline

1. We will introduce Laplace Transforms
2. Discuss their importance
3. Properties and definitions
4. Use them to solve ODEs
5. Understand partial fraction expansion
6. Transfer Functions... *They're Imaginary*...
7. Examples, Examples!

– Reading material: Dorf & Bishop, Sections 2.4 & 2.5

Laplace Transform: Basic Definition

- **Laplace Transform:** takes a function of t (time) to a function of a complex variable s (frequency)
- Given a function in time $(t \geq 0)$, $f(t)$, we want to apply this transformation:

$$\boxed{F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt}$$

- Above integral might be undefined for some s
- **Abscissa of convergence:** smallest σ such that for all s with $Re(s) \geq \sigma$, the integral above converges

Example 1: Step, Ramp, and Exponential Signals

What is the LT of $f(t) = 5 \ \forall t \geq 0$*?*

What is the LT of $f(t) = 2t \ \forall t \geq 0$*?*

What is the LT of $f(t) = e^{-at} \ \forall t \geq 0$*?*

Laplace Trasform Table

- Integration can sometimes be tedious
- And we are often too lazy to do it
- Always look at the Table and compare to what you have
- Sometimes, you have to tweak your function to fit with the given transforms

Laplace Trasform Table — 1

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$

Laplace Trasform Table — 2

19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Laplace Trasnform Properties — Linearity

- Laplace transform, by definition, is a linear mapping (transformation)
- In other words:

$$\boxed{\mathcal{L}\left[\alpha_1 f_1(t) + \alpha_2 f_2(t)\right] = \alpha_1 F_1(s) + \alpha_2 F_2(s)}$$

- Can you prove it? It's so easy
- **Proof:**
- Example: $\mathcal{L}\left[5 \cdot 1^{+}(t) + 2e^{-2}(t)\right] = ?$

Laplace Trasnform Properties — Differentiation

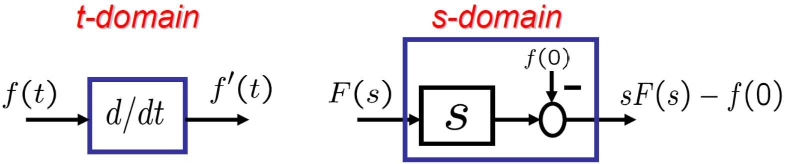
- What is the Laplace transform of a derivative of a function, $f'(t)$?

$$\boxed{\mathcal{L}\left[f'(t)\right] = sF(s) - f(0)}$$

- $f(0)$ is the initial conditions of the function $f(t)$ at $t = 0$
- Can you prove it? It's easy — you need to know integration by parts
- Example: if $f(t) = \cos(2t)$, what is $\mathcal{L}\left[f'(t)\right]$?
- Higher order differentiation:

$$\boxed{\mathcal{L}\left[f^{(n)}(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

- Illustration

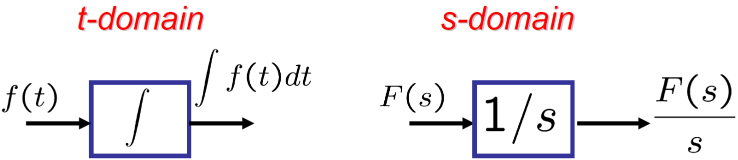


Laplace Trasnform Properties — Integration

- What is the Laplace transform of an integral of a function?

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}}$$

- Can you prove it? Integration by parts, again!
- **Proof:**
- Illustration



Laplace Trasnform Properties — Final Value Theorem

- Consider $F(s) = \dfrac{N(s)}{D(s)}$
- Poles of $F(s)$: roots$(D(s))$
- Zeros of $F(s)$: roots$(N(s))$

* **Final Value Theorem:**

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \ \textbf{ if all poles of sF(s) are in LHP}}$$

- Example 1:

$$F(s) = \frac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{5}{s^2+s+2} = \frac{5}{2}$$

- Example 2:

$$F(s) = \frac{4}{s^2+4} \Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{4s}{s^2+4} = 0 \ \textbf{ WRONG!!!}$$

FVL Example

- With all zero initial conditions for $y(t)$ and $u(t)$, system is governed by this second order ODE:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2\dot{u}(t) + u(t)$$

- Using FVT, find $\lim_{t\to\infty} y(t)$ if $u(t) = 1$
- **Solution:**

Laplace Trasnform Properties — Intial Value Theorem

- Consider $F(s) = \dfrac{N(s)}{D(s)}$

* **Initial Value Theorem:**

$$\boxed{\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) \text{ if the limit exists}}$$

- Note: In this theorem, it does not matter if pole location is in LHP or not
- Example 1:

$$F(s) = \frac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{5}{s^2+s+2} = 0$$

- Example 2:

$$F(s) = \frac{4}{s^2+4} \Rightarrow \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{4s}{s^2+4} = 0$$

Laplace Trasnform Properties — Convolution

- We all hate convolutions, right?
- What is convolution anyway?

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau)\cdot f_2(\tau)\, d\tau = \int_0^t f_2(t-\tau)\cdot f_1(\tau)\, d\tau$$

- What is the Laplace transform of $f_1 * f_2$?

$$\boxed{\mathcal{L}[f_1(t) * f_2(t)] = \mathcal{L}[f_1(t)]\cdot\mathcal{L}[f_2(t)] = F_1(s)\cdot F_2(s)}$$

- Note: $F_1(s)F_2(s) \neq \mathcal{L}[f_1(t)f_2(t)]$
- Laplace transform of convoluted functions is a smart way to run away from doing convolutions

Inverse Laplace Transform

- Given a function in time $(t \geq 0)$, $f(t)$, Laplace Transform is defined as:

$$\boxed{F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt}$$

- What if I have $F(s)$? Can I obtain $f(t)$ back? Yes.You.Can!
- Inverse Laplace Transform:

$$\boxed{f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}dt}$$

- The above integral is often very hard to solve, so it's easier to look at the table and figure things out
- Alternative: partial fraction expansion

What are Transfer Functions?

* **TFs**: *a mathematical representation to describe relationship between inputs and outputs of the physics of a system, i.e., of the differential equations that govern the motion of bodies, for example*

- **Input**: always defined as $u(t)$—called control action
- **Output**: always defined as $y(t)$—called measurement or sensor data
- TF relates the derivatives of $u(t)$ and $y(t)$
- Why is that important? Well, think of $\sum F = ma$
- F above is the input (exerted forces), and the output is the acceleration, a
- Give me the equations, please...

Construction of Transfer Functions

- For linear systems, we can often represent the system dynamics through an nth order ordinary differential equation (ODE):

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \cdots + a_0 y(t) =$$
$$u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + b_{m-2}u^{(m-2)}(t) + \cdots + b_0 u(t)$$

- The $y^{(k)}$ notation means we're taking the kth derivative of $y(t)$
- Typically, $m > n$
- Given that ODE description, we can take the LT (assuming zero initial conditions for all signals):

$$\boxed{F(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}}$$

What are Transfer Functions?

- Given this TF:

$$\boxed{F(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}}$$

- For a given control signal $u(t)$ or $U(s)$, we can find the output of the system, $y(t)$, or $Y(s)$
- But to do that, we need to take the inverse Laplace transform
- We can do that using partial fraction expansion
- **Remember:** *TFs are imaginary, $Y(s)$ often means nothing*

Partial Fraction Expansion

- **Objective 1:** find the inverse Laplace transform of $F(s)$ $(f(t) = \mathcal{L}^{-1}[F(s)])$ given that

$$\boxed{F(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}, \quad n \geq m}$$

- For linear systems, $N(s)$ and $D(s)$ **are always polynomials**
- Typically, order$(D(s)) >$ order$(N(s))$, i.e., $n \geq m$
- **Objective 1.1:** write $F(s) = \dfrac{N(s)}{D(s)}$ in terms of known expressions from the LTs table
- Three major cases—roots of $D(s)$ are: (A) distinct, (B) equal, (C) complex

Case A — Distinct $D(s)$ Roots

- Distinct roots for $D(s)$ means that we can write $F(s)$ as:

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- Since $n \geq m$, the above form can simply be written as:

$$F(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_n}{s-p_n}$$

- How will this form help me find $f(t)$?
- Well, that's easy: $f(t) = \mathcal{L}^{-1}[F(s)] = c_1 e^{p_1 t} + \cdots + c_n e^{p_n t}$
- Assume that p_i's are given
- Task: find the so-called *residues* c_i's for poles p_i's
- **Solution:** $c_i = (s-p_i)F(s)\Big|_{s=p_i}$

Case A — Example

- Find $f(t)$ if $F(s) = \dfrac{1}{s^2+5s+6}$
- Roots of s^2+5s+6 are: $p_{1,2} = -3, -2$
- Hence, $F(s) = \dfrac{1}{(s+3)(s+2)} = \dfrac{c_1}{s+3} + \dfrac{c_2}{s+2}$
- Using the method in the previous slide:

$$c_1 = (s+3)F(s)\Big|_{s=-3} = -1$$

$$c_2 = (s+2)F(s)\Big|_{s=-2} = 1$$

- Thus, $F(s) = \dfrac{1}{(s+3)(s+2)} = \dfrac{-1}{s+3} + \dfrac{1}{s+2}$
- Can you find $f(t)$ now? Of course you can

Case B — Repeated Roots

- For this case, assume that $D(s)$ includes $(s-p_1)^k$, where k is the multiplicity of pole p_1
- Find $f(t)$ if $F(s) = \dfrac{1}{(s+1)(s+2)^2} = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{(s+2)^2}$
- To find A, C, use the Case A method:

$$A = (s+1)F(s)\Big|_{s=-1} = 1, \quad C = (s+2)^2 F(s)\Big|_{s=-2} = -1$$

- To find B, substitute for $s = RandomNumber$ where $RandomNumber \neq -1, -2$
- Set $s = 0$, then $B = -1$
- Given A, B, C find $f(t)$ using the table

Case C — Imaginary Roots

- What if roots are imaginary? Remember that complex roots come in complex conjugates
- Consider that $F(s) = \dfrac{1}{s(s^2+2s+2)}$
- Roots of s^2+2s+2 are complex conjugates since $\Delta = 4-8 = -4 < 0$
- We can write

$$F(s) = \frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

- Find A using the Case A method: $A = 0.5$
- How can we find B, C? Many options, but basically, you have to solve a system of two equations, two unknowns
- **Solution:** $B = -0.5, C = -1$

Case C — Imaginary Roots (Cont'd)

- We now have:

$$F(s) = \frac{1}{s(s^2+2s+2)} = \frac{0.5}{s} + \frac{-0.5s-1}{s^2+2s+2}$$

- What is $f(t)$?
- From the LT table:

$$\boxed{\mathcal{L}^{-1}[\frac{b}{(s-a)^2+b^2}] = e^{at}\sin(bt), \mathcal{L}^{-1}[\frac{s-a}{(s-a)^2+b^2}] = e^{at}\cos(bt)}$$

- Second part of $F(s)$ can be written as:

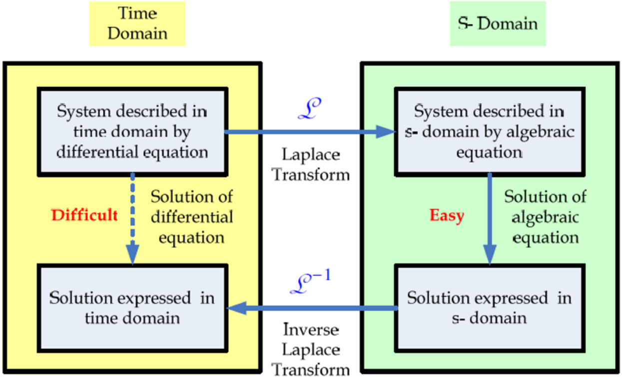
$$\frac{-0.5s-1}{s^2+2s+2} = -0.5\frac{s+2}{s^2+2s+2} = -0.5\frac{(s+1)+1}{(s+1)^2+1^2}$$

$$-0.5\frac{(s+1)}{(s+1)^2+1^2} - 0.5\frac{1}{(s+1)^2+1^2}$$

- We can now go back to the ILT table and find $f(t)$
- $\mathcal{L}^{-1}[\frac{1}{(s+1)^2+1^2}] = e^{-t}\sin(t), \mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1^2}] = e^{-t}\cos(t)$

Why? Why Not?

- Solving ODEs using time-domain techniques can be very challenging
- Laplace transforms and PFEs offer an easy way to solve ODEs
- Basic idea is as follows:



Example 1

- Solve this ODE, i.e., find $y(t)$:

$$\ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1$$

- Remember this property:

$$\boxed{\mathcal{L}\left[f^{(n)}(t)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)}$$

- **Solution:**
  1. Take Laplace Transform
  2. Find $Y(s)$
  3. Apply PFE for $Y(s)$—find the residues
  4. Use the ILT to find $y(t)$
  5. Answer: $y(t) = 1.5e^{t} - 0.5e^{-t} - t$

Example 2

- Solve this ODE, i.e., find $y(t)$:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 5, \quad y(0) = -1, \dot{y}(0) = 2$$

- **Solution:**
  1. Take Laplace Transform
  2. Find $Y(s)$
  3. Apply PFE for $Y(s)$—find the residues
  4. Use the ILT to find $y(t)$
  5. Answer: $y(t) = 1.5e^{-2t} - 5e^{-t} + 2.5$

MATLAB Demo

- Most of what we learned today can be easily tested on MATLAB
- Let's start with the basics
- First, you need to understand the language of MATLAB
- Symbolic toolbox: provides functions for solving and manipulating symbolic math expressions
- Example: `syms x y`—for LTs, define `syms s t`
- To find the Laplace transform, use the command `laplace`
- To find the inverse Laplace transform, use the command `ilaplace`

MATLAB Examples

- `syms s t`
- `ans1=laplace(t*exp(3*t))`
- `iplace(ans1)`
- `ans2=laplace(t*t*cos(5*t)*exp(3*t))`
- `iplace(ans2)`
- Let's try the PFE, command: `residue`
- `[R,P,K] = residue(B,A)`

Where are we now?

In this module, we learned:

- How to analytically compute Laplace transforms
- How to be lazy and look at the table—unless you are a genius
- Final and initial value theorems, because computing analytical limits is too mainstream
- Inverse Laplace transforms and PFEs
- Solving ODEs using LTs
- Where are we now?

Modeling:

- ~~**Laplace Transforms**~~
- ~~**Transfer Functions**~~
- ~~**Solution of ODEs**~~
- Modeling of Systems
- Block Diagrams
- Linearization

Analysis:

- 1^{st} & 2^{nd} Order Systems
  - Time Response
  - Transient & Steady State
- Frequency Response
- Bode Plots
- RH Criterion
- Stability Analysis

Design:

- Root-Locus
- Modern Control
- State-Space
- MIMO System Properties

Questions And Suggestions?

Thank You!

Please visit

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IFF you want to know more ☺