# Module 03
# Modeling of Dynamical Systems

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**EE 3413: Analysis and Desgin of Control Systems**

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# Module 3 Outline

1. Physical laws and equations
2. Transfer function model
3. Model of electrical systems
4. Model of mechanical systems
5. Examples

– Reading material: Dorf & Bishop, Section 2.3

# Physical Laws and Models

- By definition, dynamical systems **are dynamic** because they change with time
- Change in the sense that their intrinsic properties evolve, vary
- Examples: coordinates of a drone, speed of a car, body temperature, concentrations of chemicals in a centrifuge
- Physicists and engineers like to represent dynamic systems with equations
- Why? Well, the answer is fairly straightforward
- Dynamic model often means a differential equations

# Physical Laws

- For many systems, it's easy to understand the physics, and hence the math behind the physics
  - Examples: circuits, motion of a cart, pendulum, suspension system
- For the majority of dynamical systems, the actual physics is complex
- Hence, it can be hard to depict the dynamics with ODEs
  - Examples: human body temperature, thermodynamics, spacecrafts
- This illustrates the needs for ***models***
- **Dynamic system model:** a mathematical description of the actual physics

# What are Transfer Functions?


* **TFs**: *a mathematical representation to describe relationship between inputs and outputs of the physics of a system, i.e., of the differential equations that govern the motion of bodies, for example*

- **Input**: always defined as $u(t)$—called control action
- **Output**: always defined as $y(t)$—called measurement or sensor data
- TF relates the derivatives of $u(t)$ and $y(t)$
- Why is that important? Well, think of $\sum F = ma$
- $F$ above is the input (exerted forces), and the output is the acceleration, $a$
- Give me the equations, please...

# Construction of Transfer Functions


- For linear systems, we can often represent the system dynamics through an $n$th order ordinary differential equation (ODE):

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \cdots + a_0 y(t) =$$

$$u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + b_{m-2}u^{(m-2)}(t) + \cdots + b_0 u(t)$$

- The $y^{(k)}$ notation means we're taking the $k$th derivative of $y(t)$
- Typically, $m > n$
- Given that ODE description, we can take the LT (assuming zero initial conditions for all signals):

$$\boxed{H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}}$$

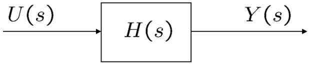
# What are Transfer Functions?


- Given this TF:

$$\boxed{H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}}$$

- For a given control signal $u(t)$ or $U(s)$, we can find the output of the system, $y(t)$, or $Y(s)$
- **Impulse response:** defined as $h(t)$—the output $y(t)$ if the input $u(t) = \delta(t)$
- **Step response:** the output $y(t)$ if the input $u(t) = 1^+(t)$
- For any input $u(t)$, the output is: $\boxed{y(t) = h(t) * u(t)}$
- But...Convolutions are nasty...Who likes them?

# TFs of Generic LTI Systems

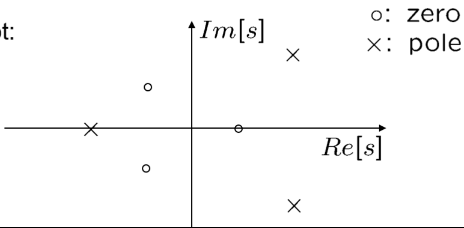



- So, we can take the Laplace transform: $\boxed{Y(s) = H(s)U(s)}$
- Typically, we can write the TF as:

$$\boxed{H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}}$$

- Roots of numerator are called the **zeros** of $H(s)$ or the system
- Roots of the denominator are called the **poles** of $H(s)$

Pole zero plot:



# Example

Given: $H(s) = \dfrac{2s+1}{s^3 - 4s^2 + 6s - 4}$

- **Zeros**: $z_1 = -0.5$
- **Poles**: solve $s^3 - 4s^2 + 6s - 4 = 0$, use MATLAB's `roots` command
  * `poles=roots[1 -4 6 -4]`; $poles = \{2, 1+j, 1-j\}$
- **Factored form:**

$$H(s) = 2\frac{s+0.5}{(s-2)(s-1-j)(s-1+j)}$$

# Analyzing Generic Physical Systems

*Seven-step algorithm:*

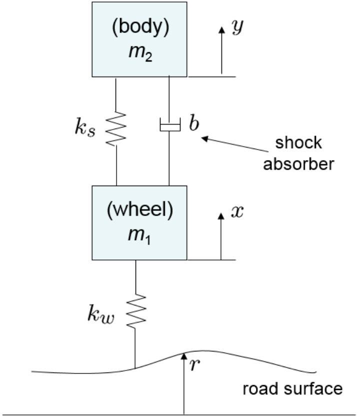
1. Identify dynamic variables, inputs ($u$), and system outputs ($y$)
2. Focus on one component, analyze the dynamics (physics) of this component
   - How? Use Newton's Equations, KVL, or thermodynamics laws...
3. After that, obtain an $n$th order **ODE**:

$$\boxed{\sum_{i=1}^{n} \alpha_i y^{(i)}(t) = \sum_{j=1}^{m} \beta_j u^{(j)}(t)}$$

4. Take the Laplace transform of that **ODE**
5. Combine the equations to eliminate internal variables
6. Write the transfer function from input to output
7. For a certain control $U(s)$, find $Y(s)$, then $y(t) = \mathcal{L}^{-1}[Y(s)]$

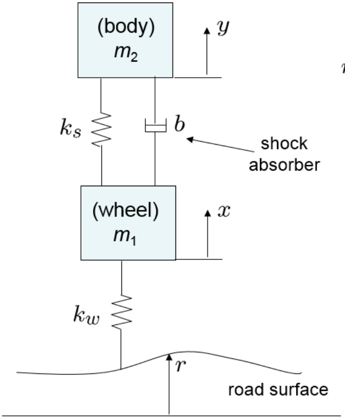
# Active Suspension Model

- Each car has 4 active suspension systems (on each wheel)
- System is nonlinear, but we consider approximation. **Objective?**
- **Input**: road altitude $r(t)$ (or $u(t)$), **Output**: car body height $y(t)$



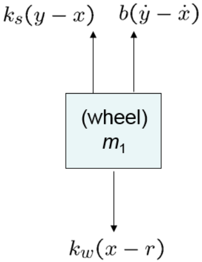
# Active Suspension Model — Equations for 1 Wheel

- We only consider one of the four systems
- System has many components, most important ones are: body ($m_2$) & wheel ($m_1$) weights

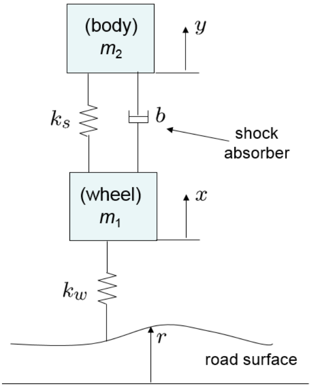


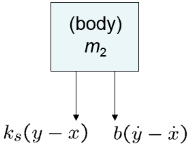

By Newton's Second Law

$$m_1\ddot{x} = k_s(y-x) + b(\dot{y}-\dot{x}) - k_w(x-r)$$



# Active Suspension Model — Equations for Car Body




By Newton's Second Law

$$m_2\ddot{y} = -k_s(y-x) - b(\dot{y}-\dot{x})$$

- We now have 2 equations depicting the car body and wheel motion
- Objective: find the TF relating output $(y(t))$ to input $(r(t))$
- What is $H(s) = \dfrac{Y(s)}{R(s)}$?

# Active Suspension Model — Transfer Function

- **Differential equations (in time):**

$$\begin{aligned} m_1\ddot{x}(t) &= k_s(y(t)-x(t)) + b(\dot{y}(t)-\dot{x}(t)) - k_w(x(t)-r(t)) \\ m_2\ddot{y}(t) &= -k_s(y(t)-x(t)) - b(\dot{y}(t)-\dot{x}(t)) \end{aligned}$$

- **Take Laplace transform given zero ICs:**

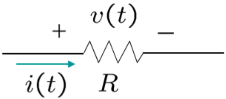
– **Solution:**

- Find $H(s) = \dfrac{Y(s)}{R(s)}$

– **Solution:**

# Basic Circuits Components

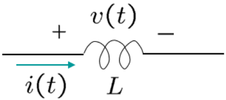
resistor




$$v(t) = Ri(t)$$

$$V(s) = RI(s) \Rightarrow \frac{V(s)}{I(s)} = R$$

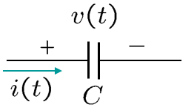
inductor




$$v(t) = L\frac{di(t)}{dt}$$

$$V(s) = LsI(s) \Rightarrow \frac{V(s)}{I(s)} = Ls$$

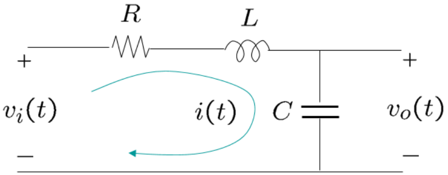
capacitor




$$i(t) = C\frac{dv(t)}{dt}$$

$$I(s) = CsV(s) \Rightarrow \frac{V(s)}{I(s)} = \frac{1}{Cs}$$

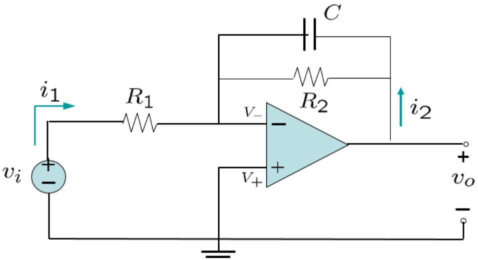
# Basic Circuits — RLCs & Op-Amps




$v_i(t)$ : input

$v_o(t)$ : output

Transfer function $\frac{V_o(s)}{V_i(s)}$

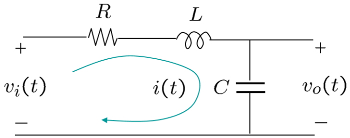



$v_i(t)$ : input

$v_o(t)$ : output

Transfer function $\frac{V_o(s)}{V_i(s)}$

# TF of an RLC Circuit — Example




**Objective: Find TF**

$v_i(t)$ : input

$v_o(t)$ : output

Transfer function $\frac{V_o(s)}{V_i(s)}$

- Apply KVL (assume zero ICs):

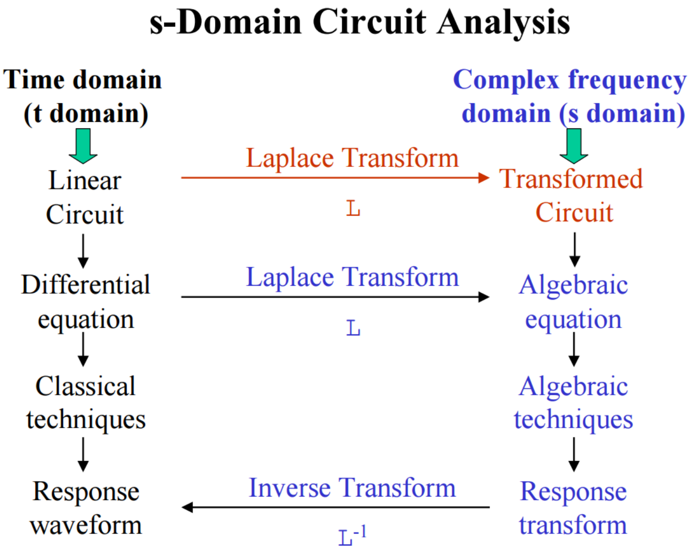
$$\begin{aligned} v_i(t) &= Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(\tau)dt \\ v_o(t) &= \frac{1}{C}\int i(\tau)dt \end{aligned}$$

- Take LT for the above differential equations:

$$\begin{aligned} V_i(s) &= RI(s) + LsI(s) + \frac{1}{Cs}I(s) \\ V_o(s) &= \frac{1}{Cs}I(s) \Rightarrow I(s) = CsV_o(s) \end{aligned}$$

$$\Rightarrow \boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

# Generic Circuit Analysis



# Dynamic Models in Nature

- Predator-prey equations are 1st order non-linear, ODEs
- Describe the dynamics of biological systems where 2 species interact
- One species as a predator and the other as prey
- Populations change through time according to these equations:

$$\dot{x}(t) = \alpha x(t) - \beta x(t)y(t)$$

$$\dot{y}(t) = \delta x(t)y(t) - \gamma y(t)$$

  - $x(t)$: # of preys (e.g., rabbits)
  - $y(t)$: # of predators (e.g., foxes)
  - $\dot{x}(t), \dot{y}(t)$: growth rates of the 2 species—function of time, $t$
  - $\alpha, \beta, \gamma, \delta$: +ve real parameters depicting the interaction of the species

# Mathematical Model

$$\dot{x}(t) = \alpha x(t) - \beta x(t)y(t)$$

$$\dot{y}(t) = \delta x(t)y(t) - \gamma y(t)$$

- Prey's population grows exponentially ($\alpha x(t)$)—why?
- Rate of predation is assumed to be proportional to the rate at which the predators and the prey meet ($\beta x(t)y(t)$)
- If either $x(t)$ or $y(t)$ is zero then there can be no predation
- $\delta x(t)y(t)$ represents the growth of the predator population
- No prey $\Rightarrow$ no food for the predator $\Rightarrow$ $y(t)$ decays
- Is there an equilibrium? What is it?

# Nonlinear Dynamical Systems

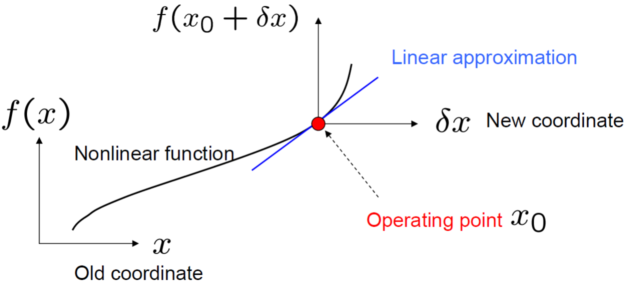
- Let's face it: most dynamical systems are **nonlinear**
- Nonlinearities can be seen in the ODEs, e.g.:

$$\dot{y}(t) + \dot{y}(t)\ddot{y}(t) + \cos(y(t)) = 2u(t) + \arctan(e^{\cos(u(t))})$$

- Examples: electromechanical systems, electronics, hydraulic systems, thermal, etc...
- *Why do we hate nonlinear systems?*
  - Well, because we cannot solve ODEs tractably if they are not linear
- I mean we can, but they're hard—and remember, we're lazy
- **Solution: linearize** nonlinear equations
- Btw...most nonlinear systems are linear for a *short period of time*
- So, it's legit to linearize for a *short period of time*

# Linearization — The Main Idea

- Linearization is one of the most important techniques in control theory
- Without it, all our analysis of nonlinear systems becomes pointless
- First, let's assume that a nonlinear system is **linearized around an operating point**
- **Operating point** is often called **equilibrium point**
- Main idea:



# Linearization — The Simple Math

- Nonlinear equation (or system): $\dot{x}(t) = f(x, u)$
- **E**quilibrium points: $u_e, x_e$
- Equilibrium **deviation** : $\delta u(t) = u(t) - u_e$, $\delta x(t) = x(t) - x_e$
- Taylor series expansion around $u_e, x_e$:

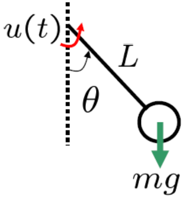
$$\boxed{\dot{x}(t) \approx f(x_e, u_e) + (\delta x(t)) \left. \frac{\partial f(x,u)}{\partial x} \right|_{x_e, u_e} + (\delta u(t)) \left. \frac{\partial f(x,u)}{\partial u} \right|_{x_e, u_e}}$$

- Hence:

$$\boxed{\delta \dot{x}(t) \approx (\delta x(t)) \left. \frac{\partial f(x,u)}{\partial x} \right|_{x_e, u_e} + (\delta u(t)) \left. \frac{\partial f(x,u)}{\partial u} \right|_{x_e, u_e}}$$

- This relationship is a linear one between $\delta x$ and $\delta u$

# Linearization — Example




- Pendulum motion:

$$f(x,u) = -\frac{g}{L}\sin(x(t)) + \frac{1}{mL^2}u(t)$$

- $x(t)$: angle $(\theta)$, $u(t)$: force
- Given **e**quilibrium points: $u_e = 0, x_e = \pi$
- Taylor series expansion around $0, \pi$:

$$\boxed{\delta f(\delta x, \delta u) \approx \frac{g}{L}\delta x(t) + \frac{1}{mL^2}\delta u(t)}$$

- This relationship is a linear one between $\delta x$ and $\delta u$: only valid in the *vicinity of the equilibrium point*

# Roadmap Revisited

*Modeling* (5-6 Weeks)

- ~~**Laplace Transforms**~~
- ~~**Transfer Functions**~~
- ~~**Solution of ODEs**~~
- ~~**Modeling of Systems**~~
- Block Diagrams
- ~~**Linearization**~~

*Analysis* (7-8 Weeks)

- 1st & 2nd Order Systems
  - Time Response
  - Transient & Steady State
- Frequency Response
- Bode Plots
- RH Criterion
- Stability Analysis

*Design* (5-6 Weeks)

- Root-Locus
- Modern Control
- State-Space
- MIMO System Properties

# Questions And Suggestions?


**Thank You!**

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**IFF** you want to know more ☺