Module 05
System Analysis & First and Second Order Dynamical Systems

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EE 3413: Analysis and Desgin of Control Systems

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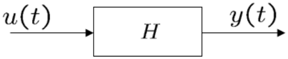
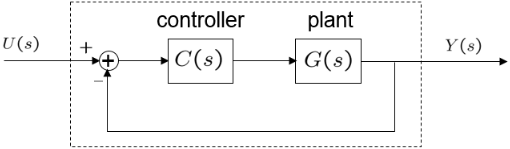
Module 5 Outline

1. General linear systems analysis
2. Responses to different test signals
3. First order systems & properties
4. Second order systems & properties
5. Reading sections: 5.1–5.5 Ogata, 5.1–5.4 Dorf and Bishop

What have we done so far?

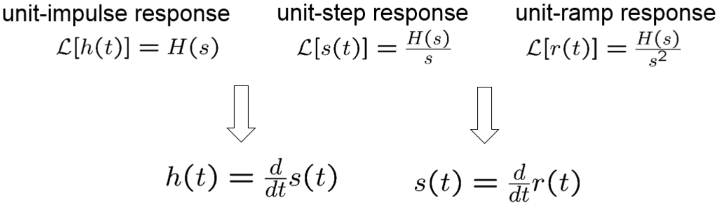
- Well...So far, we know how to model a dynamical system
- + Reduce blocks to a single transfer function
- Module's goal: **analyze + characterize** input-output behavior
- Simple idea: want to know how your system is performing?
- Yes! Well, excite it with different test inputs ⇒ study the response





Test Inputs

1. Impulse input: $u(t) = \delta(t)$, Output: impulse-response, $y_i(t) = \mathcal{L}^{-1}[H(s)] = h(t)$
2. Step input: $u(t) = 1^+(t)$, Output: step-response, $y_s(t) = ?$
   - Step input characterizes system's ability to track sudden input changes
3. Ramp input: $u(t) = t$, Output: ramp-response, $y_r(t) = ?$
   - Ramp input characterizes system's ability to track varying input
   - Why are these important? How is this useful? Relationship between them:



unit-impulse response

$$\mathcal{L}[h(t)] = H(s)$$

unit-step response

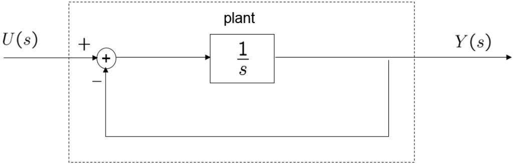
$$\mathcal{L}[s(t)] = \frac{H(s)}{s}$$

unit-ramp response

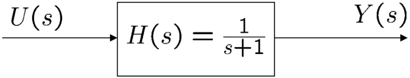
$$\mathcal{L}[r(t)] = \frac{H(s)}{s^2}$$

$$h(t) = \frac{d}{dt}s(t) \qquad s(t) = \frac{d}{dt}r(t)$$

Example



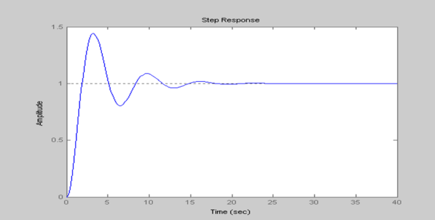
- First, we find the overall transfer function, $H(s)$
- Solution:



Input	Response
$u(t) = \delta(t)$	$h(t) = e^{-t} \cdot 1(t)$
$u(t) = 1(t)$	$s(t) = (1 - e^{-t}) \cdot 1(t)$
$u(t) = t \cdot 1(t)$	$r(t) = (t - 1 + e^{-t}) \cdot 1(t)$

Transient Vs. Steady State Responses

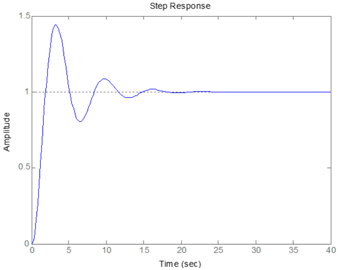
- Any output for linear system is decomposed of: $\boxed{y(t) = y_{ss}(t) + y_{tr}(t)}$
- $y_{ss}(t)$: stead-state response — signifies the system's ability to *eventually* track input signals after few seconds
- $y_{tr}(t)$: transient response — path the output took to reach SS
- Overly oscillatory $y_{tr}(t)$ is usually bad for systems. Why?
- Slow transient response is typically undesirable. Why?
- Example:



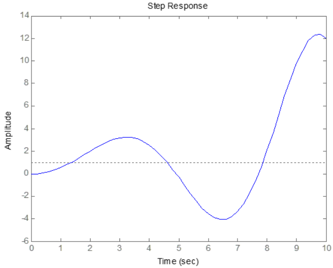
Stable Vs. Unstable Systems? How to Characeterize?

- **Stable system: step response converges to a finite value** OR
- **Impulse response converges to...?**
- **Unstable system: step response output doesn't converge**
- Example:

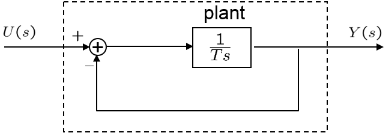
Stable

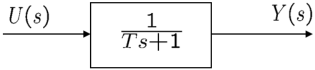


Unstable



First Order Systems





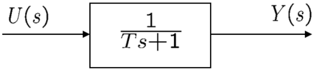
- What's the meaning of first order systems?
- They're characterized by this TF:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ts+1}, \quad T = \text{time constant}$$

- Can we derive the ODE related to the input and output?
- What happens if $T < 0$? $T > 0$?
- What happens when T varies? For $T > 0$:
  - Larger $T \Rightarrow$ slower decay (larger time-constant)
  - Smaller $T \Rightarrow$ faster decay (smaller time-constant)

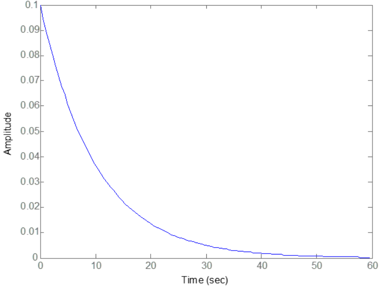
First Order System: Stability Analysis & Impulse Response

- For smaller T, system will go to zero faster
- Plots show the impulse response, $h(t)$

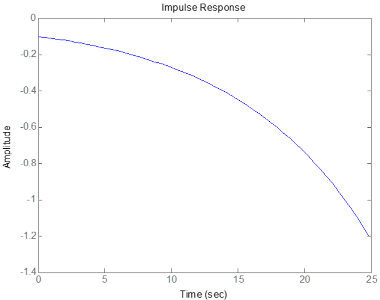


System is stable if $T>0$, and unstable if $T<0$

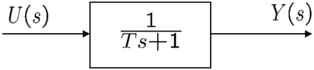
$T=10$



$T=-10$



Step Response


$U(s)$

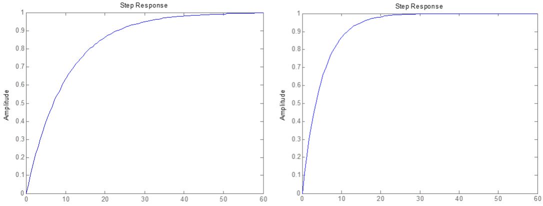
$\frac{1}{Ts+1}$

$Y(s)$

- What is the step response of the FOS?

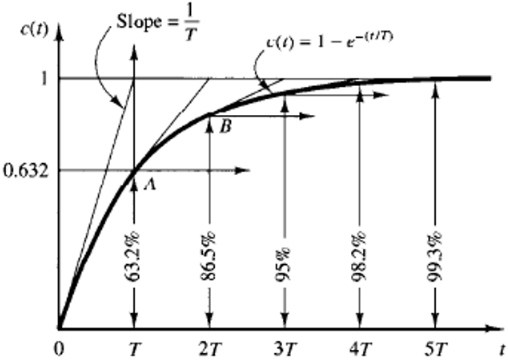
$$Y_{step}(s) = H(s)U(s) = \frac{1}{Ts+1}\frac{1}{s} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}} \Rightarrow y_{step}(t) = 1 - e^{\frac{-t}{T}}$$

- Similar to impulse response, smaller $T \Rightarrow$ faster response
- Example:

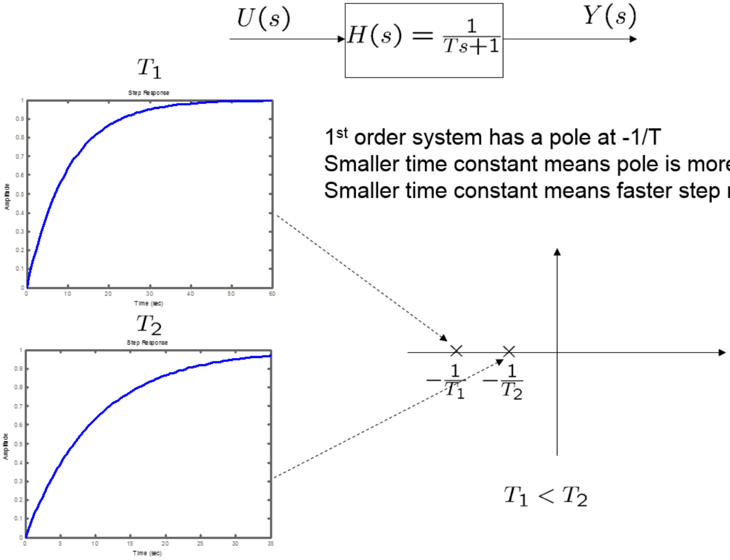

$T = 10$ $T = 5$

Time Constant and Step Response

- What happens if $t = T$, i.e., $t = 1$ time constant?
- Answer: $y_{step}(t = T) = 1 - e^{\frac{-T}{T}} = 1 - e^{-1} = 0.632$
- How many time constants do we need to reach steady-state (SS)?
- Solution: after $t \geq 5T$, we reach 99.3% of SS



Effect of Poles on Step Response

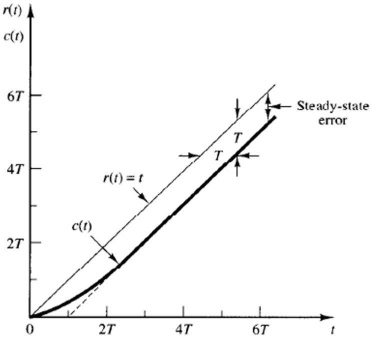


1^{st} order system has a pole at -1/T
Smaller time constant means pole is more to the left
Smaller time constant means faster step responses

Ramp Response of FOSs

- So far, we've done impulse and step responses of FOSs
- Now: ramp response. Again, why are we doing this?
- What is the impulse response of the FOS?

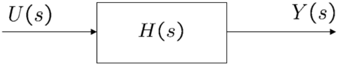
$$Y_{ramp}(s) = H(s)U(s) = \frac{1}{Ts+1}\frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} - \frac{T^2}{Ts+1} \Rightarrow y_{ramp}(t) = t - T + Te^{\frac{-t}{T}}$$



Important Remarks on FOSs

- Location of the pole (i.e., $p = -1/T$) determines the response of FOSs
- Transient will settle down (i.e., stable) if p is in the LHP
- If the pole is further on the LHP, transients will settle down faster
- Why are there no oscillations for step response of FOSs?
- I'll give you brownie points if you guess :)

SOSs: Introduction and Definition



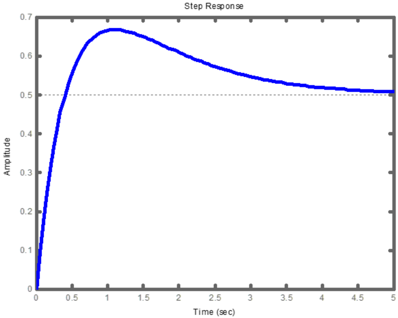
- Generic TF of SOSs:

$$H(s) = \frac{b_0 s^2 + b_1 s + b_2}{a_0 s^2 + a_1 s + a_2}$$

- Most important thing for SOSs: the location of the poles of $H(s)$
- SOS is **called stable** if **all poles are in the LHP**

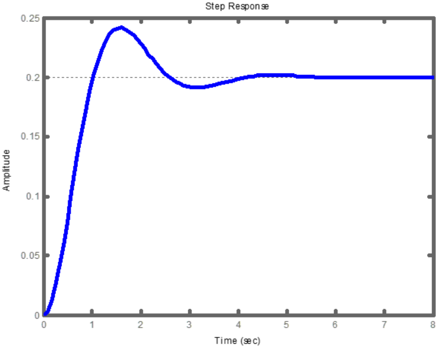
Step Response of Stable SOS

- Example: $H(s) = \frac{2s+1}{s^2+3s+2}$ — poles: $p_1 = -2, p_2 = -1$
- What's $y_{step}(t)$? We should know how to obtain that by now
- $y_{step}(t) = e^{-t} - 1.5e^{-2t} + 0.5$
- Poles p_1 and p_2 contribute to a term in $y_{step}(t)$
- However, since both poles are stable, step response converges to a SS value $= 0.5$ — notice the so-called *overshoot*



What happens if the poles are stable, but complex?

- Another motivating example: $H(s) = \dfrac{1}{s^2 + 2s + 5}$
- Poles: $p_{1,2} = -1 \pm 2i$ — stable poles (LHP), complex conjugates
- Step response: $y_{step}(t) = 0.2 - 0.2e^{-t}\cos(2t) + 0.1e^{-t}\sin(2t)$
- Sines and cosines $\Rightarrow$ oscillations, right? What's the SS value?
- Step response:



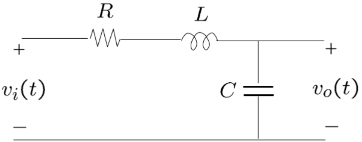
More Common Standard Form of SOSs

- The most common standard form of SOSs:

$$H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- This form: (a) represents only one family of SOSs, (b) denominator polynomial has **+ve coefficients**, (c) $H(0) = 1$
- Definitions: (a) $\omega_n \equiv$ **undamped natural frequency**, (b) $\zeta \equiv$ **damping ratio**
- $\omega_n > 0$, $\zeta > 0$

SOS Example: finding ω_n and ζ



$v_i(t)$: input
$v_o(t)$: output

- Recall this circuit example from Module 3
- What was the TF? $H(s) = \dfrac{1}{LCs^2 + RCs + 1}$
- This is not in the standard form (previous slide)
- In standard form: $H(s) = \dfrac{1/LC}{s^2 + \underbrace{R/L}_{=2\zeta\omega_n} \cdot s + \underbrace{1/LC}_{=\omega_n^2}}$
- Hence: $\omega_n = \sqrt{1/LC}$, $\zeta = \dfrac{R}{2\sqrt{\dfrac{L}{C}}}$

Poles of SOSs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Poles:

$$\boxed{p_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}}$$

- SOS has two poles — how many cases to consider? Three cases:
  - **Underdamped case:** Two **complex conjugate** poles $\Rightarrow 0 < \zeta < 1$
  - **Critically damped case:** Two **identical real** poles $\Rightarrow \zeta = 1$
  - **Overdamped case:** Two **distinct real** poles $\Rightarrow \zeta > 1$

Case 1 — Underdampled System, $0 < \zeta < 1$

$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ has two complex poles

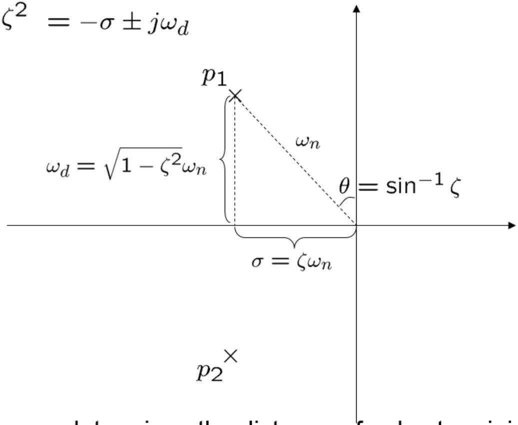
$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$

- where

$$\sigma = \zeta\omega_n$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

“dampled natural frequency”



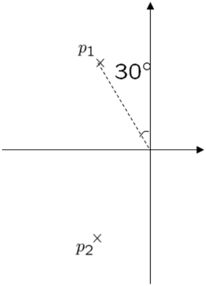
- Angle $\theta = \sin^{-1}\zeta$
- As ζ increases from 0 to 1:

θ increases from 0 to 90 degree

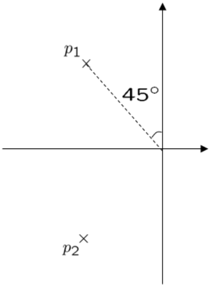
- Undampled natural frequency ω_n determines the distance of poles to origin
- Damping ratio ζ determines the angle θ

Case 1 — Underdampled System, Examples

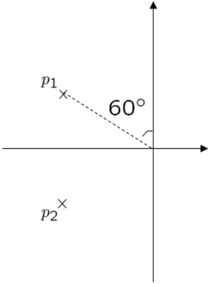
$\zeta = 0.5$



$\zeta = 0.707$



$\zeta = 0.866$

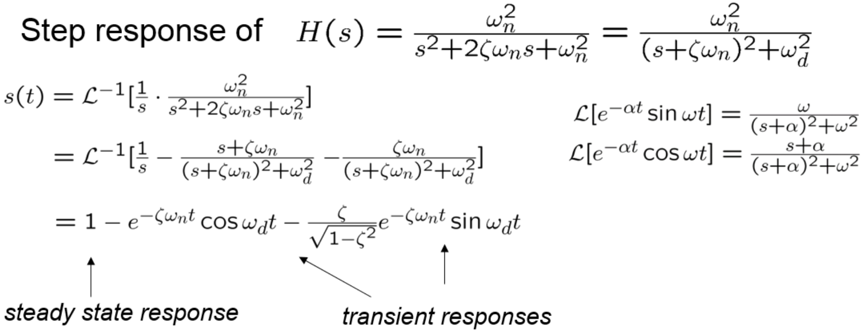


Case 1 — Underdampled System, Step Response

- We can easily obtain the step response given Case 1 ($0 < \zeta < 1$)
- Since we have complex poles, $p = -\sigma + j\omega_d$, taking the inverse Laplace transform for $1/(s+p)$ would yield exponentially decaying sines and cosines:

$$e^{pt} = e^{(-\sigma+\omega_d)t} = e^{-\sigma t}\left(\cos(\omega_d t) + j\sin(\omega_d t)\right)$$

- What are the transients and SS components?



Step response of $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_d^2}$

$$s(t) = \mathcal{L}^{-1}[\tfrac{1}{s} \cdot \tfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}]$$

$$= \mathcal{L}^{-1}[\tfrac{1}{s} - \tfrac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \tfrac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}]$$

$$= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$\mathcal{L}[e^{-\alpha t}\sin\omega t] = \frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$\mathcal{L}[e^{-\alpha t}\cos\omega t] = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

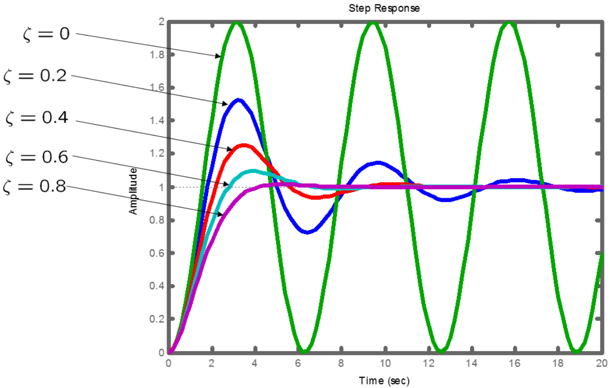
steady state response

transient responses

Case 1 — Underdampled System Step Response

- Here, we change ζ, while ω_n is constant for an underdamped system
- Remember that

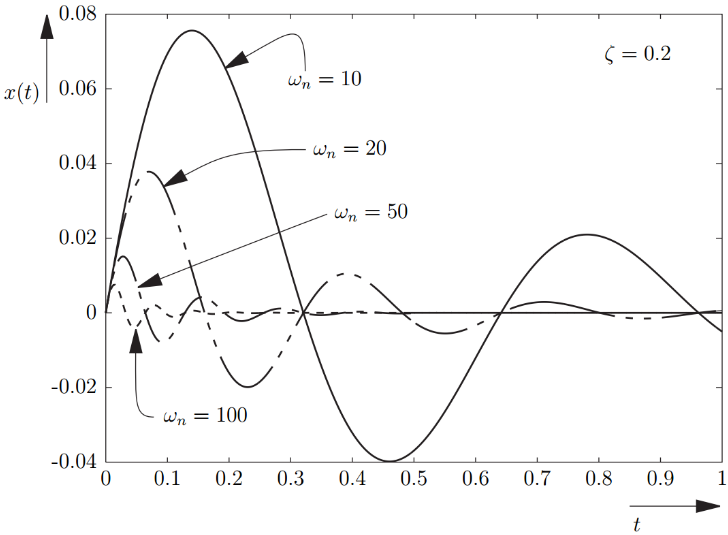
$$\boxed{s(t) = y_{step}(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t)}$$



Case 1 — Underdampled System, Important Remarks

- As we saw in the previous plot for different ζ for underdamped case, we have overshoot and oscillation
- Real part of the poles ($\sigma = \zeta\omega_n$) determines transient amplitude decaying rate
- Imaginary part of the poles (ω_d) determines transient oscillation frequency
- For a given undamped system, as $\zeta \nearrow$:
  - Angle $\theta \nearrow$, poles shift more to the left, $\omega_d \searrow$
  - Overshoot $\searrow$
- What happens if we $\nearrow \omega_n$ and fix ζ?

Fixing ζ and Increasing ω_n



Case 2 — Critically Damped System, $\zeta = 1$

- This case is not that interesting — not as much as Case 1
- Why? Cz we have 2 identical real poles at the same location (LHP):

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

- Poles: $p_{1,2} = -\omega_n$
- Step response? $y_{step}(t) = \mathcal{L}^{-1}[\frac{1}{s} \cdot \frac{\omega_n^2}{(s+\omega_n)^2}] = 1 - e^{-\omega_n t}(1 + \omega_n t)$
- How did we get this from the step response of underdamped case?

$$y_{step}^{under}(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t)$$

- Well, this can be obtained by letting ζ approach 1 and use the limit of $\sin(\alpha x)/x = \alpha$ as $x \to 0$:

$$\lim_{\zeta \to 1} \frac{\sin(\omega_d t)}{\sqrt{1-\zeta^2}} = \lim_{\zeta \to 1} \frac{\sin(\omega_n \sqrt{1-\zeta^2}\, t)}{\sqrt{1-\zeta^2}} = \omega_n t$$

Case 3 — Overdamped System, $\zeta > 1$

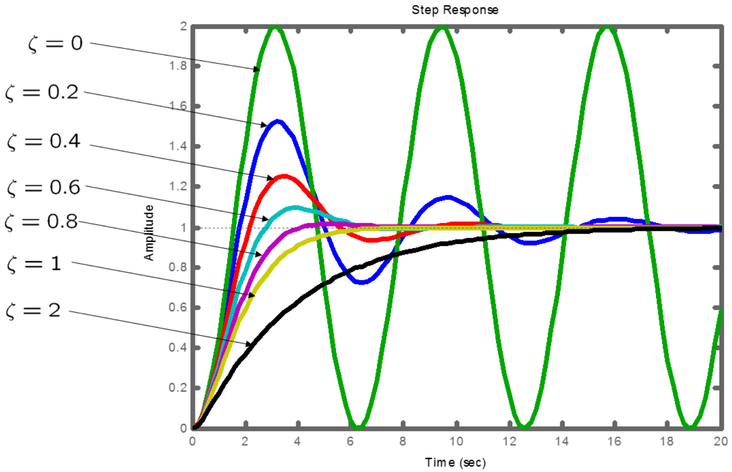
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Also, not a very interesting case...Actually, a very boring one
- Poles: **distinct real poles**, $p_{1,2} = -(\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$
- Step response:

$$y_{step}^{over}(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{p_1 t}}{p_1} - \frac{e^{p_2 t}}{p_2}\right)$$

- Can approximate overdamped second order systems as first order systems?
- Yes. How? Dominant poles...

Step Response for Different ζ



- For $\zeta \geq 1$, system response mimics what?

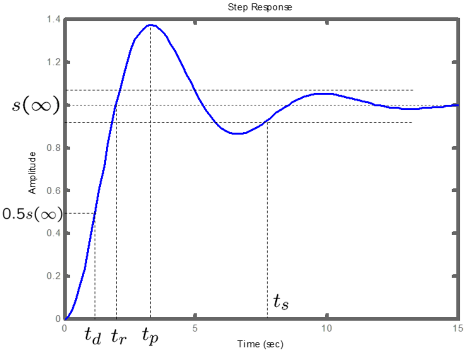
VERY Important Remarks on SOSs

- Overdamped system is slow in responding to inputs — takes time to reach SS
- That depends on how far the poles are in the LHP
- For systems without oscillations, which one responds faster to inputs? In other words, which one reaches SS faster?
  - Answer: critically damped system, $\zeta = 1$ — see previous plot
- Underdamped systems with $0.5 \leq \zeta \leq 0.8$ get close to the final value more rapidly than critically dampled or overdampled system, without incurring too large overshoot
- How can we obtain impulse or ramp response of second order systems?
  - Answer: by differentiation and integration, respectively.

Time Specs of Systems

1. t_d: **delay time** — time for $y_{step}(t)$ to reach half of $y_{step}(\infty)$
2. t_r: **rise time** — time for $y_{step}(t)$ to reach first $y_{step}(\infty)$
3. t_p: **peak time** — time for $y_{step}(t)$ to reach first peak
4. M_p: **maximum overshoot** — $M_p = \dfrac{y_{step}(t_p) - y_{step}(\infty)}{y_{step}(\infty)}$
5. t_s: **settling time** — time for $y_{step}(t)$ to settle within a range of 2% 5% of $y_{step}(\infty)$

A typical step response $s(t)= y_{step}(t)$



Time Specs of Systems — 2

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Given ζ and ω_n, can we determine the time-specs in terms of them?
- I mean can we have an equations that relate the two?
- We can, yes...We'll focus on the underdamped case as three time-specs aren't defined for critically and overdamped systems
- Step response, revisited:

$$s(t) = y_{step}(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t)$$
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \beta)$$

- $\beta = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$

Closed form Solution of Time Specs

- **Delay time:** find the smallest positive solution to $y_{step}(t_d) = 0.5$
- **Rise time:** smallest positive solution of $y_{step}(t_r) = 1 \Rightarrow$

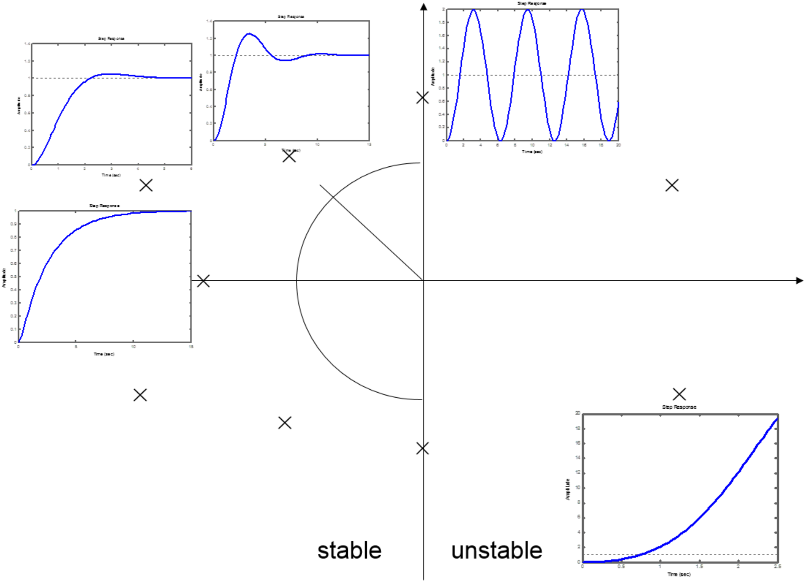
$$\boxed{t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \tan^{-1} \dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\sqrt{1-\zeta^2}\omega_n}}$$

- **Peak time:** smallest positive solution to $y'_{step}(t_p) = 0$: $\boxed{t_p = \dfrac{\pi}{\omega_d}}$
- **Maximum overshoot:**

$$M_p = \frac{y_{step}(t_p) - y_{step}(\infty)}{y_{step}(\infty)} = y_{step}(t_p) - 1 \Rightarrow \boxed{M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}}$$

- **Settling time:** $t_s \approx \dfrac{4}{\zeta\omega_n}$ (2% criteria), $t_s \approx \dfrac{3}{\zeta\omega_n}$ (5% criteria),
- $t_p \searrow$ with ω_n; the smaller the ζ, the larger the M_p

Effect of Pole Locations on Responses of SOSs



Where Are We Now?

Modeling
(5-6 Weeks)

- ~~**Laplace Transforms**~~
- ~~**Transfer Functions**~~
- ~~**Solution of ODEs**~~
- ~~**Modeling of Systems**~~
- ~~**Block Diagrams**~~
- ~~**Linearization**~~

Analysis
(7-8 Weeks)

- ~~**1st & 2nd Order Systems**~~
  - ~~**Time Response**~~
  - ~~**Transient & Steady State**~~
- Frequency Response
- Bode Plots
- RH Criterion
- Stability Analysis

Design
(5-6 Weeks)

- Root-Locus
- Modern Control
- State-Space
- MIMO System Properties

Questions And Suggestions?

Thank You!

Please visit

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IFF you want to know more ☺