Module 07
Control Systems Design & Analysis via Root-Locus Method

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EE 3413: Analysis and Desgin of Control Systems

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ELECTRICAL & COMPUTER ENGINEERING UTSA ROADRUNNERS

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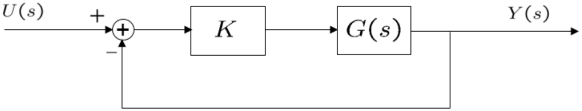
Module 7 Outline

1. Introduction to root-locus
2. Root-locus plots and examples
3. Root-locus steps and rules
4. Design of feedback systems via root-locus
5. More examples
6. Reading sections: 6.1–6.5 Ogata, 7.1–7.4 Dorf and Bishop

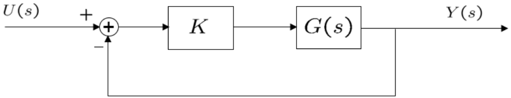
Introduction to Root-Locus (RL)

- Previously, we learned that CLTF poles determine everything
- Poles determine stability, transients, steady-state error
- What if we change one of the constants?
- If we change one parameter, how will the CL poles change?
  - Will the system go unstable? Will the transients change?
- **Example**: remember the finding the K example in Module 06?
  - Finding the best K for this system...We'll learn how CLTF change if we change K



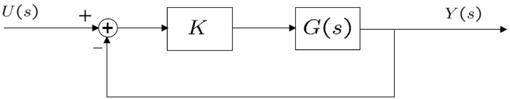
$$G(s) = \frac{1}{s(s^2+s+1)(s+2)}$$

RL Preliminaries

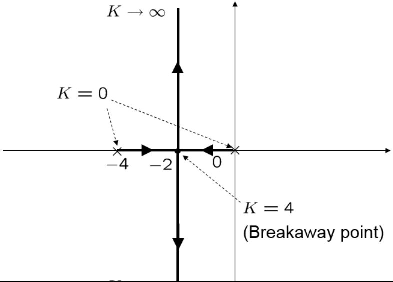


- **Root-Locus:** graph showing how poles vary as $K \nearrow$ from $0 \to \infty$
- We'll start with the above system (unity feedback), but RL is general for different systems
- RL: a graph showing closed-loop poles as 1 system parameter changes
- **Definition:** the characteristic polynomial (CP) of CLTF is $1 + KG(s)H(s) = 0$
- This equation determines the RL
- For the above system, $H(s) = 1$
- RL method can be generalized for different style of systems

RL Example

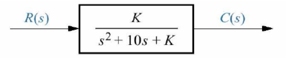


- Example: let's plot the RL for $G(s) = 1/(s(s+4))$
- CP: $1 + K\frac{1}{s(s+4)} = 0 \Rightarrow s^2 + 4s + K = 0 \Rightarrow p_{1,2} = -2 \pm \sqrt{4-K}$
- For $K = 0$: $p_{1,2} = 0, -4$; $K = 4$: $p_{1,2} = -2, -2$; $K > 1$: $p_{1,2} =$ complex #



RL Another Example

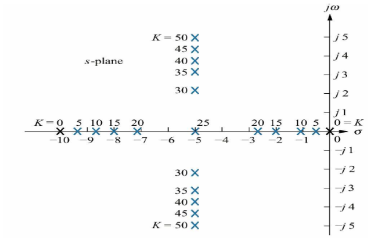
- We can always obtain the RL by taking different values for parameter K and then evaluating the corresponding CLTF poles (roots of CP)
- Another example:



The location of poles as a function of K can be calculated as

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

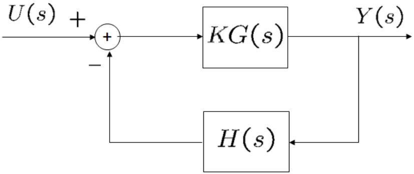
The corresponding root locus can be drawn



RL Plot Intro

- The previous examples are relatively easy
- Roots of CP easy to derive in terms of K — not the case for 3rd and higher order systems
- **Objective:** have a systematic way of drawing the RL for **any system**
- RL has many branches, is a weird-looking plot (which makes sense)
- RL helps infer design values for gains of control systems
- Tedious solution: have a table for different Ks and corresponding CLTF poles (as in the previous example)
- Then, draw the root locus by connecting the points
- We'll learn a more systematic approach that's easy to execute

The Rules



Defs. OLTF $= G(s)H(s)$; $n_p, n_z =$ number of poles, zeros of OLTF

Rule 1 RL is symmetric with respect to **the real-axis—**remember that

Rule 2 RL has n branches, $n = n_p$

Rule 3 Mark poles (n_p) and zeros (n_z) of $G(s)H(s)$ with 'x' and 'o'

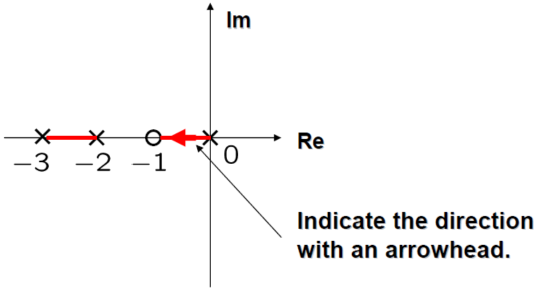
Rule 4 Each branch starts at OLTF poles ($K = 0$), ends at OLTF zeros or at infinity ($K = \infty$)

Rule 5 RL has branches on x-axis. These branches exist on real axis portions where the **total # of poles + zeros** to the right is an odd #

Example 2, Executing Rules 1–5

$$G(s) = \frac{s+1}{s(s+2)(s+3)}, H(s) = 1 \Rightarrow \text{ OLTF} = K\frac{s+1}{s(s+2)(s+3)}$$

- Rules 1–5:
  - $n_p = 3, n_z = 1$, # of Branches $= 3$, plot so far:



  - Hence, we have to figure out the shape **of only 1 more branch**

RL Rules 6 and 7: Angles of Asymptotoes

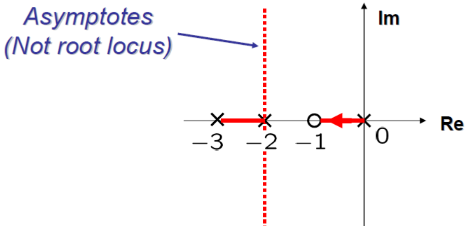
Rule 6 Asymptotes angles: RL branches ending at OL zeros at ∞ approach the asymptotic lines with angles:

$$\phi_q = \frac{(1+2q)180}{n_p - n_z} \deg, \forall q = 0, 1, 2, \ldots, n_p - n_z - 1$$

Rule 7 Real-axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} Re(p_i) - \sum_{j=1}^{n_z} Re(z_j)}{n_p - n_z}$$

- Solution of Rules 6 and 7:
- $\phi_q = 90$ and 270 degrees, $\sigma_A = \frac{0+(-2)+(-3)-(-1)}{2} = -2$



Rule 8: Break-away & Break-in Points

$$1 + KG(s)H(s) = 0 \Rightarrow K(s) = \frac{-1}{G(s)H(s)} \Rightarrow \frac{dK(s)}{ds} = -\frac{d}{ds}\left[\frac{1}{G(s)H(s)}\right]$$

Rule 8-1 RL branches intersect the real-axis at points where K is at an extremum for real values of s. We find the breakaway points by finding solutions (i.e., s^* solutions) to:

$$\frac{dK(s)}{ds} = 0 = -\frac{d}{ds}\left[\frac{1}{G(s)H(s)}\right] = 0 \Rightarrow \frac{d}{ds}[G(s)H(s)] = 0 \Rightarrow \text{ obtain } s^*$$

Rule 8-2 After finding s^* solutions (you can have a few), check whether the corresponding $K(s^*) = \dfrac{-1}{G(s^*)H(s^*)} = K^*$ is **real positive #**

Rule 8-3 **Breakaway pt.:** K^*_{max} (-ve $K''(s^*)$), **Break-in pt:** K^*_{min} (+ve $K''(s^*)$)

Solution for Rule 8

- Recall that $K(s) = \dfrac{-1}{G(s)H(s)}$
- For the given $G(s), H(s)$, we get:

$$\frac{d}{ds}\left[G(s)H(s)\right] = 0 \Rightarrow$$

  – Recall derivative rule for fractions (quotient rule)

- Hence:

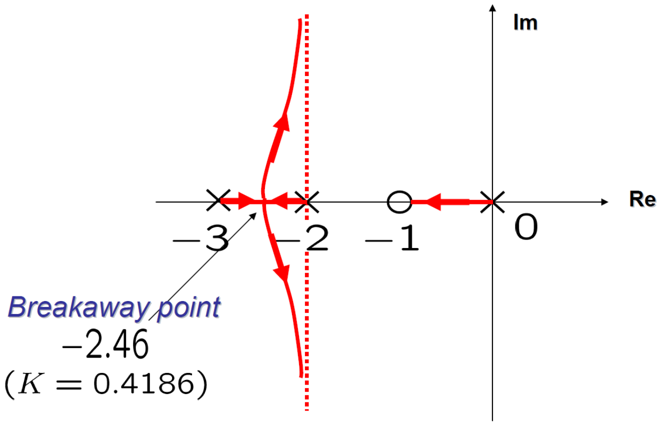
$$\frac{d}{ds}\left[G(s)H(s)\right] = 0 \Rightarrow \frac{d}{ds}\left(\frac{s+1}{s(s+2)(s+3)}\right) = \frac{-2s^3 - 8s^2 - 10s - 6}{\left(s(s+2)(s+3)\right)^2} = 0$$

Rule 8-1 $s^* = $ **-2.46**, $-0.76 \pm 0.79i$ (3 solutions, since numerator is 3rd degree polynomial, now we check which s^* yields $K > 0$, next)

Rule 8-2 $K(s^*) = \dfrac{-1}{G(s^*)H(s^*)} = \mathbf{0.418}, 1.79 \pm 4.27 \Rightarrow K^* = \mathbf{0.418}$

Rule 8-3 $K''(s^* = -2.46) < 0 \Rightarrow$ breakaway point

Solution for Rule 8



Two More Rules (None Apply in This Example)

Rule 9 **Angle of Departure (AoD):** defined as the angle from a complex pole or Angle of Arrival (AoA) at a complex zero:

$$\text{AoD from a complex pole}: \ \phi_p = 180 - \sum_i \angle p_i + \sum_j \angle z_j$$

$$\text{AoA at a complex zero}: \ \phi_z = 180 + \sum_i \angle p_i - \sum_j \angle z_j$$

- $\sum_i \angle p_i$ is the sum of all angles of vectors to a complex pole in question from other poles, $\sum_j \angle z_j$ is the sum of all angles of vectors to a complex pole in question from other zeros
- '$\angle$' denotes the angle of a complex number

Rule 10 Determine whether the RL crosses the imaginary y-axis by setting:

$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i$$

and finding the ω and K that solves the above equation. The value of ω you get is the frequency at which the RL crosses the imaginary y-axis and the K you get is the associated gain for the controller

Example 3 — RL for $G(s) = \dfrac{1}{s(s+1)(s+5)}, H(s) = 1$

Defs. $n_p = 3$, $n_z = 0$, $\text{OLTF} = \dfrac{1}{s(s+1)(s+5)}$

Rule 1 RL is symmetric with respect to the real-axis—OK!

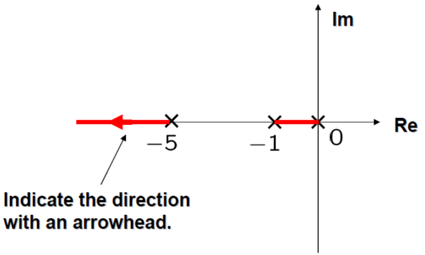
Rule 2 RL has 3 branches

Rule 3 Marking poles (no zeros)

Rule 4 Branches start @ OLTF p's (K=0), ends @ OLTF z's or @ $K = \infty$

Rule 5 Draw branches on the x-axis via the odd total number of z's and p's

Solution ... so far:



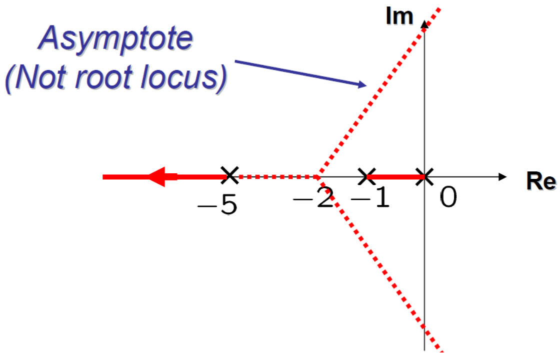
Example 3 (Cont'd)

Rule 6 Asymptotes angles:

$$\phi_q = \frac{(1+2q)180}{3} \quad \forall q = 0,1,2 \;\; || = \{60, 180, 300\}\, \text{deg}$$

Rule 7 Real axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} Re(p_i) - \sum_{j=1}^{n_z} Re(z_j)}{n_p - n_z} = -2$$



Example 3 (Cont'd)

Rule 8-1 Find breakaway/breaking points:

$$\frac{dK(s)}{ds} = 0 = -\frac{d}{ds}\left[\frac{1}{G(s)H(s)}\right] = 0 \Rightarrow \frac{d}{ds}[G(s)H(s)] = 0 \Rightarrow \text{ obtain } s^*$$

$$\Rightarrow 3s^2 + 12s + 5 = 0 \Rightarrow s^*_{1,2} = -2 \pm \frac{\sqrt{21}}{3} = -0.47, -3.3$$

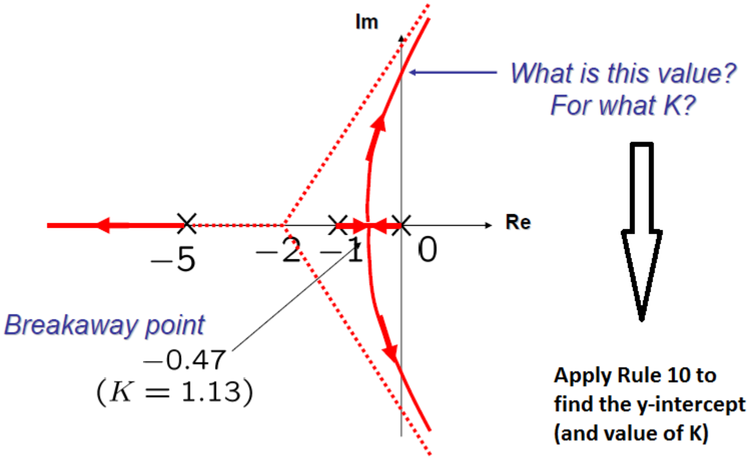
- Notice that s_2^* does not belong to the RL

Rule 8-2 Find corresponding values for K^*:

$$K_1^* = 1.13\ (s_1^* \approx -0.47), K_2^* = -13.1\ (s_2^* \approx -3.3) \Rightarrow \boxed{K^* = 1.13} > 0$$

Rule 8-3 s_1^* is **a breakaway point** since $K''(s_1^*) < 0$

Solution for Rule 8



Apply Rule 10 to find the y-intercept (and value of K)

Rules 9 and 10

Rule 9 Undefined since we do not have complex poles or zeroes

Rule 10 Determine where the RL crosses the imaginary y-axis by setting:

$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i$$

$$\Rightarrow 1 + \frac{K}{j\omega(j\omega + 1)(j\omega + 5)} = 0 + 0j$$

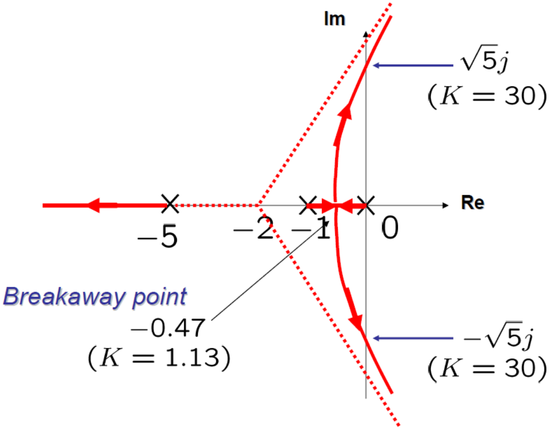
$$\Rightarrow -6\omega^2 + K + j(-\omega^3 + 5\omega) = 0 + 0j$$

$$\Rightarrow K = 6\omega^2, \;\; \omega(-\omega^2 + 5) = 0$$

$$\Rightarrow \boxed{\omega^* = \pm j\sqrt{5}, K^* = 30} \text{ — the y-axis crossings}$$

Overall Solution

Always remember that the RL plot of any system is the way poles change for different values of K or any other parameter. Hence, for a stable system, we would want $0 < K < 30$. *We could have checked that via the Routh array too.*



Example 4 — RL for $G(s) = \frac{s+2}{s^2+2s+3}, H(s) = 1$

Defs. $n_p = 2,\ n_z = 1,\ \text{OLTF} = \frac{s+2}{s^2+2s+3}$

P's, Z's $p_{1,2} = -1 \pm j\sqrt{2},\ z_1 = -2$

Rule 1 RL is symmetric with respect to the real-axis—OK!

Rule 2 RL has **only** 2 branches

Rule 3 Marking poles and zeros

Rule 4 Branches start @ OLTF p's (K=0), ends @ OLTF z's or @ $K = \infty$

Rule 5 Draw branches on the x-axis via the odd total number of z's and p's

Solution ... So far

Example 4 (Cont'd)

Rule 6 Asymptotes angles:

$$\phi_q = \frac{(1+2q)180}{1} \quad \forall q = 0 \;\; || = 180\,\text{deg}$$

Rule 7 Real axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} Re(p_i) - \sum_{j=1}^{n_z} Re(z_j)}{n_p - n_z} = 0 \quad \text{— no actual intersection}$$

Solution

Example 4 (Cont'd)

$$K(s) = -\frac{s^2+2s+3}{s+2}$$

Rule 8-1 Find breakaway/breaking points:

$$\frac{dK(s)}{ds} = 0 \Longrightarrow \text{ obtain } s^*$$

$$\Rightarrow s^2+4s+1=0 \Rightarrow s^*_{1,2} = -3.73, -0.26$$

– Notice that s_2^* does not belong to the RL so most likely it's not a solution

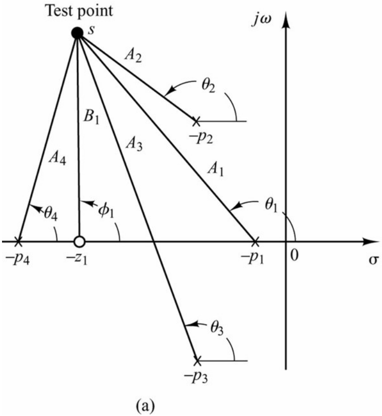
Rule 8-2 Find corresponding values for K^*:

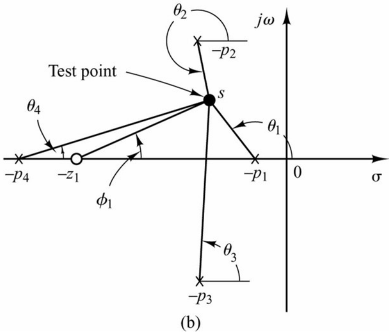
$$K_1^* = 5.46\ (s_1^* = -3.73), K_2^* = -1.46,\ (s_2^* = -0.26) \Rightarrow \boxed{K^* = 5.46} > 0$$

Rule 8-3 s_1^* is **a break-in point** since $K''(s_1^*) > 0$

Rule 9

Figures show angle measurements from open-loop poles & open-loop zero to test point s





Rule 9 **Angle of departure** from $p_1 = 180 - 90 + 55 = 145$, from $p_2 = 180 - 270 + 245 = -145$, no angle of arrival since we have no complex zeros

Rule 10

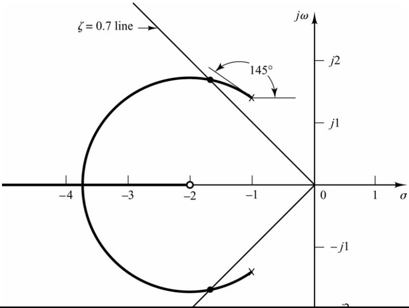
Rule 10 Determine where the RL crosses the imaginary y-axis by setting:

$$1 + KG(s = j\omega)H(s = j\omega) = 0 + 0i \Rightarrow 1 + \frac{K(j\omega + 2)}{(j\omega)^2 + 2j\omega + 3} = 0 + 0j$$

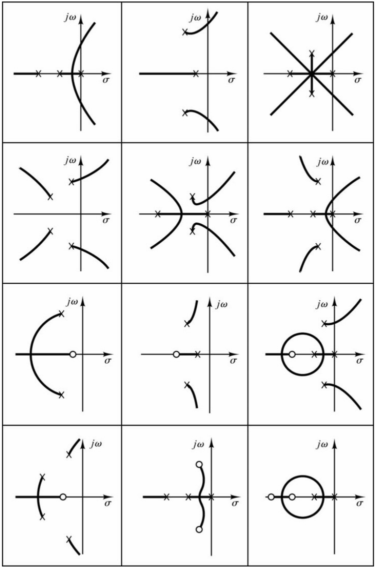
$$\Rightarrow \omega^2 - 3 + j(-2\omega - k\omega - 2k) = 0 + 0j \Rightarrow \omega = \pm\sqrt{3}$$

– This value for ω cannot be on the root locus, hence no crossings on the y-axis

- Overall solution



Generic RL Plots



Example 5 — RL for $G(s) = \dfrac{1}{s(s^2+4s+5)}, H(s) = 1$

Defs. $n_p = 3$, $n_z = 0$, OLTF $= \dfrac{1}{s(s^2+4s+5)}$

Rule 1 RL is symmetric with respect to the real-axis—OK!

Rule 2 RL has 3 branches

Rule 3 Marking poles (no zeros)

Rule 4 Branches start @ OLTF p's (K=0), ends @ OLTF z's or @ ∞

Rule 5 Draw branches on the x-axis via the odd total number of z's and p's

Solution ... so far

Example 5 (Cont'd)

Rule 6 Asymptotes angles:

$$\phi_q = \frac{(1+2q)180}{3} \quad \forall q = 0, 1, 2 \; || = \{60, 180, 300\}\,\text{deg}$$

Rule 7 Real axis intercept of asymptotes:

$$\sigma_A = \frac{\sum_{i=1}^{n_p} Re(p_i) - \sum_{j=1}^{n_z} Re(z_j)}{n_p - n_z} = -\frac{0+2+2}{3} = -1.333$$

Example 5 (Cont'd)

Rule 8-1 Find breakaway/breaking points:

$$K(s) = -(s^3 + 4s^2 + 5s)dK(s)/ds = 0 \Rightarrow s^*_{1,2} = -1, -1.667$$

– Note that both solutions are on the root locus

Rule 8-2 Find corresponding values for K^*:

$$K^*_{1,2} = 2, 1.85 > 0$$

Rule 8-3 s_1^* **is a breakaway point**, s_2^* **is a break-in point** (second derivative rule of $K(s)$)

Rule 9 Angle of departure from the complex poles:

$$\phi_{p_1} = 190 - 90 - 153.43 = -63.43\,\text{deg}, \phi_{p_2} = 63.43\,\text{deg}$$

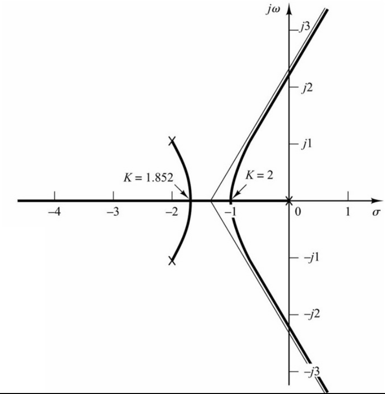
Rule 10 + Overall Solution

Rule 10 Determine where the RL crosses the imaginary y-axis by setting:

$$1+KG(s=j\omega)H(s=j\omega)=0+0i \Rightarrow (j\omega)^3+4(j\omega)^2+5(j\omega)+K=0+0j$$

$$\Rightarrow \boxed{\omega^*=\pm\sqrt{5}, K^*=20} \text{ — the y-axis crossings}$$

Solution



RL Module Summary

- Root-Locus: a graphical representation of the CLTF poles (can tell stability from that) as a function of one design parameter
- This design parameter — in this Module — was the gain K
- RL gives more information about the poles in comparison with other stability criterion
- RHSC: gives only ranges for K for stable systems
- RL: gives a plot of how the poles are changing as a function of K
- You have to practice a lot of RL plots if you wanna master the RL arts

Questions And Suggestions?

Thank You!

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