Module 06
Stability of Dynamical Systems

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October 10, 2017

Stability of CT LTV Systems

The following CT LTI system without inputs

$$\dot{x}(t) = A(t)x(t), \;\; x(t) \in \mathbb{R}^n$$

has an equilibrium at $x_e = 0$.

Asymptotic Stability

The above system is asymptotically stable at $x_e = 0$ if its solution $x(t)$ starting **from any initial condition** $x(t_0)$ **satisfies**

$$x(t) \to 0, \text{as} \;\; t \to \infty$$

Exponential Stability

The above system is exponentially stable at $x_e = 0$ if its solution $x(t)$ starting **from any initial condition** $x(t_0)$ **satisfies**

$$\|x(t)\| \leq Ke^{-rt}\|x(t_0)\|, \quad \forall t \geq t_0$$

for some positive constants K and r.

Example 1

Consider the following TV LTI system from Homework 4:

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{t+1} & 0 \\ -\frac{1}{t+1} & 0 \end{bmatrix} x(t)$$

- Recall that the solution to this system is

$$x(t) = \phi(t,0)x(0) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ -\frac{t}{t+1} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1/5 \end{bmatrix} = \begin{bmatrix} \frac{1}{t+1} \\ -\frac{t}{t+1} - 1/5 \end{bmatrix}$$

- Is this system asymptotically stable?
- **Solution:** it's not, since the states do not go to zero for any initial conditions

Stability of CTLTI Systems

- For this CT LTI system

$$\dot{x}(t) = Ax(t)$$

  the solution $x(t) = e^{At}x(t_0)$ is a linear combination of the modes of the system
- In other words, $x(t)$ is a linear combinations of $p(t)e^{\lambda_i t}$
- $p(t)$ is a polynomial of t
- Why does that make sense? Well...

Stability of LTI Systems

The following theorems are equivalent:

- The LTI system is asymptotically stable
- The LTI system is exponentially stable
- All eigenvalues of A are in the open left half of the complex plane

Marginal Stability

Definition of Marginal Stability

The CT LTI system $\dot{x}(t) = Ax(t)$ is called marginally stable if **both of these statements are true:**

- All eigenvalues of A are in the closed[a] LHP
- There are some eigenvalues of A on the $j\omega$-axis, and all the Jordan blocks associated with such eigenvalues have size one

[a] A closed set can be defined as a set which contains all its limit points.

For marginally stable systems:

- Starting from any initial conditions, the solution $x(t)$ will neither converge to zero nor diverge to infinity
- State solutions will converge (not necessarily to zero) **only if all evalues at the $j\omega$ axis are zero**
- Can you justify these findings?
- **From now on: stability means asymptotic stability**

Example 2

- Consider the CT LTI system with $A = \begin{bmatrix} -12 & -4 \\ -2 & -1 \end{bmatrix}$
- This system has evalues $\lambda_1 = -12.68, \lambda_2 = -0.31$
- The two evalues are in the LHP
- Hence, the system is asymptotically stable

Unstable LTI Systems

Definition of Instability

The CT LTI system $\dot{x}(t) = Ax(t)$ is unstable if **either of these statements is true**

- A has an eigenvalue (or eigenvalues) in the open RHP
- A has an eigenvalues on the $j\omega$-axis whose at least one Jordan block has size greater than one

*This means that the state solutions will diverge to infinity

Example 3

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

- Is this system stable?
- From Homework 3, the state solution is (we solved for initial conditions $x(1)$):

$$x(t) = e^{A(t-1)}x(1) = \begin{bmatrix} 1 \\ 2t-1 \\ 6t^2 - 6t + 1 \end{bmatrix}$$

- Clearly, this system is unstable
- Eigenvalues are all equal to zero, and the size of Jordan blocks is three (greater than 1)

Stability of LTV Systems

- We talked about asymptotic and exponential stability
- These concepts are easy to verify for LTI systems
- What about CT LTV systems? What are the evalues of $A(t)$?
- You cannot often answer this question
- Solution? **Find the STM**
- Recall that $x(t) = \phi(t, t_0)x(t_0)$ for LTV systems
- **System is asymptotically stable iff** $\phi(t, t_0) \to 0$ **as** $t \to \infty$
- **System is exponentially stable iff there exist positive constants** C, r **such that**

$$\|\phi(t, t_0)\| \leq Ce^{-rt}$$

  for all $t \geq t_0$
- For LTV systems, asymptotic stability is **not** equivalent to exponential stability

Example 2

Consider the following TV LTI system from Homework 4:

$$\dot{x}(t) = \begin{bmatrix} -1+\cos(t) & 0 \\ & -2+\sin(t) \end{bmatrix} x(t)$$

- The state transition matrix is:

$$\phi(t, t_0) = \begin{bmatrix} e^{-(t-t_0)+\sin(t)-\sin(t_0)} & 0 \\ 0 & e^{-2(t-t_0)+\cos(t_0)-\sin(t)} \end{bmatrix}$$

- Is this system exponentially stable?
- **Solution:** We'll have to prove that $\|x(t)\| \leq Ke^{-rt}\|x(t_0)\|, \quad \forall t \geq t_0$ and basically obtain K and r
- Note that: $\|\phi(t,t_0)x(t_0)\| \leq \|\phi(t,t_0)\|\|x(t_0)\|$ and

$$|e^{-(t-t_0)+\sin(t)-\sin(t_0)}| = |e^{-(t-t_0)}| \cdot |e^{\sin(t)-\sin(t_0)}| \leq e^2 e^{-(t-t_0)}$$

$$|e^{-2(t-t_0)+\cos(t_0)-\cos(t)}| = |e^{-2(t-t_0)}| \cdot |e^{\cos(t_0)-\cos(t)}| \leq e^2 e^{-2(t-t_0)}$$

- Hence, we can extract K and r given the norm of $\phi(t, t_0)$: $K = e^2 \cdot e^{t_0}$, $r = 1$

Stability of DT LTV Systems

Consider the following DT LTI system

$$x(k+1) = A(k)x(k), \;\; x(k) \in \mathbb{R}^n$$

Asymptotic Stability

The above system is asymptotically stable at time k_0 its solution $x[k]$ starting from any initial condition $x(k_0)$ at time k_0 **satisfies:**

$$x(k) \to 0, \text{as} \;\; k \to \infty$$

Exponential Stability

The above system is exponentially stable at time k_0 its solution $x[k]$ starting from any initial condition $x(k_0)$ at time k_0 **satisfies:**

$$\|x(k)\| \leq Kr^{k-k_0}\|x(k_0)\|, \quad \forall k = k_0, k_0+1, k_0+2, \dots$$

for some constants $K > 0$ and $0 \leq r < 1$.

Stability of DT LTI Systems

- For this DT LTI system

$$x(k+1) = Ax(k)$$

the following theorems are equivalent:

Stability of DT LTI Systems

- The DT LTI system is asymptotically stable
- The DT LTI system is exponentially stable
- All eigenvalues of A are inside the open unit disk of the complex plane

Example 4

$$x(k+1) = \begin{bmatrix} 0.5 & 0.3 \\ 0 & -0.4 \end{bmatrix} x(k)$$

- This system has two eigenvalues:

$$\lambda_1 = 0.5, \lambda_2 = -0.4$$

- Both are inside the unit disk, hence the system is stable

Marginal Stability, Instability of DT LTI Systems

Definition of Marginal Stability

The DT LTI system $x(k+1) = Ax(k)$ is called marginally stable if **both of these statements are true:**

- All eigenvalues of A are inside the closed unit disk
- There are some eigenvalues of A on the unit circle, and all the Jordan blocks associated with such eigenvalues have size one

Definition of Instability

The DT LTI system $x(k+1) = Ax(k)$ is unstable if **either of these statements is true**

- A has an eigenvalue (or eigenvalues) outside the closed unit disk
- A has an eigenvalues on the unit circle whose at least one Jordan block has size greater than one

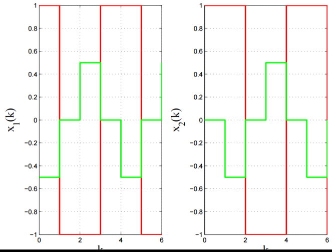
Example 5

$$x(k+1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(k), x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

- This system has two eigenvalues: $\lambda_1 = j, \lambda_2 = -j$
- These evalues are located on the boundaries of the unit disk
- The state solution is given:

$$x_1(k) = x_{10}\cos(0.5k\pi) + x_{20}\sin(0.5k\pi)$$
$$x_2(k) = x_{20}\cos(0.5k\pi) + x_{10}\sin(0.5k\pi)$$

- For any $x(0)$, this system will be marginally stable



Stability of DT LTV Systems

- For DT LTV systems, asymptotic stability is **not** equivalent to exponential stability
- Recall that $x(k) = \phi(k, k_0)x(k_0)$ for DT LTV systems $(x(k+1) = A(k)x(k))$
- **DT LTV system is asymptotically stable iff $\phi(k, k_0) \to 0$ as $k \to \infty$**
- **DT LTV system is exponentially stable iff there exist $C > 0$ and $0 \leq r < 1$ such that**

$$\|\phi(k, k_0)\| \leq Cr^{k-k_0}$$

for all $k \geq k_0$

Summary

system		*continuous-time*	*discrete-time*
asympt. stable	$\forall i = 1,\ldots,n$	$\Re(\lambda_i) < 0$	$\lvert\lambda_i\rvert < 1$
unstable	$\exists i$ such that	$\Re(\lambda_i) > 0$	$\lvert\lambda_i\rvert > 1$
stable	$\forall i,\ldots,n$	$\Re(\lambda_i) \leq 0$	$\lvert\lambda_i\rvert \leq 1$
	and $\forall \lambda_i$ such that algebraic = geometric mult.	$\Re(\lambda_i) = 0$	$\lvert\lambda_i\rvert = 1$

- In the above table, "stable" means **marginally stable**

Aleksandr Mikhailovich Lyapunov (1857—1918)

Intro to Lyap Stability

- Lyapunov methods: very general methods to prove (or disprove) stability of nonlinear systems
- *Lyapunov's stability theory is the single most powerful method in stability analysis of nonlinear systems.*
- Consider a nonlinear system: $\dot{x}(t) = f(x)$
  - A point x_{eq} is an equilirbium point if $f(x_{eq}) = 0$
  - Can always consider that $x_0 = 0$; if not, you can shift coordinates
- Any equilibrium point is:
  - **Stable in the sense of Lyapunov:** if (arbitrarily) small deviations from the equilibrium result in trajectories that stay (arbitrarily) close to the equilibrium for all t
  - **Asymptotically stable:** if small deviations from the equilibrium are eventually *forgotten* and the system returns asymptotically to the equilibrium point
  - **Exponentially stable:** if the system is asymptotically stable, and the convergence to the equilibrium point is **fast**

The Math

The equilibrium point is

- **Stable in the sense of Lyapunov (ISL)** (or simply **stable**) if for each $\epsilon \geq 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$||x(0)|| < \delta \;\Rightarrow\; ||x(t)|| \leq \epsilon, \;\; \forall t \geq 0$$

- **Asymptotically stable** if there exists $\delta > 0$ such that

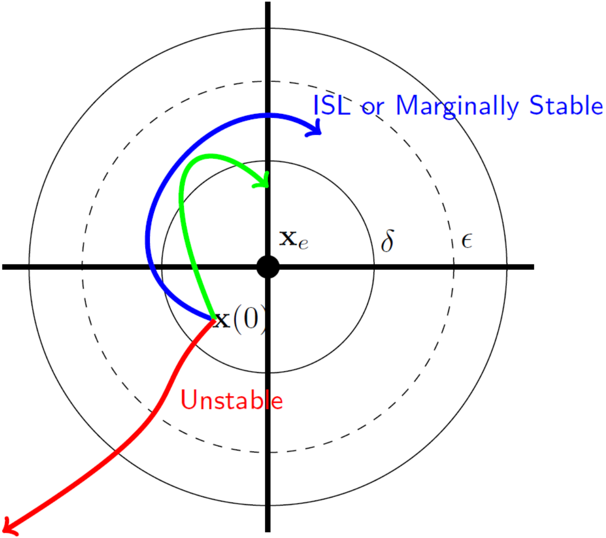
$$||x(0)|| < \delta \;\Rightarrow\; \lim_{t\to\infty} x(t) = 0$$

- **Exponentially stable** if there exist $\{\delta, \alpha, \beta\} > 0$ such that

$$||x(0)|| < \delta \;\Rightarrow\; ||x(t)|| \leq \beta e^{-\alpha t}, \;\; \forall t \geq 0$$

- **Unstable** if not stable

Stability of nonlinear systems



More on Lyap Stability

- How do we analyze the stability of an equilibrium point **locally**?
- Well, for nonlinear systems we can find all equilibrium points (previous modules)
- We can obtain the linearized dynamics $\dot{\tilde{x}}(t) = A_{eq}^{(i)}\tilde{x}(t)$ for all equilibria $i = 1, 2, \ldots$
- You can then find the eigenvalues of $A_{eq}^{(i)}$: if all are negative, then that particular equilibrium point is **locally stable**
- This method is called **Lyapunov's first method**
- How about global conclusion for $\dot{x}(t) = f(x(t))$?
- You'll have to study **Lyapunov Function** that give you insights on the global stability properties of nonlinear systems
- We can't cover these in this class

Simple Example

- Analyze the stability of this system

$$\dot{x}(t) = \frac{2}{1+x(t)} - x(t)$$

- This system has two equilibrium points:

$$x_{eq}^{(1)} = 1, \; x_{eq}^{(2)} = -2$$

- Analyze stability of each point
- Example 2: the inverted pendulum in the previous lecture

Questions And Suggestions?

Thank You!

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IFF you want to know more ☺