Model Predictive Control
Part I – Introduction

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Main Idea

Objective:

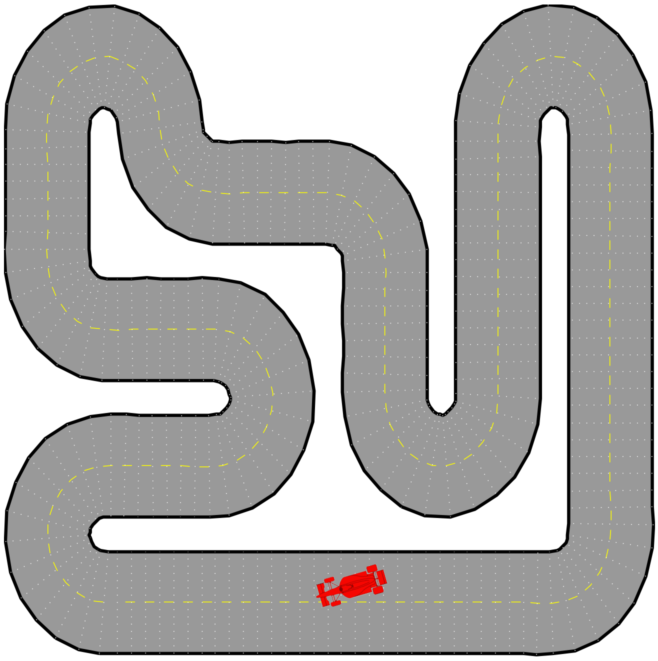
- Minimize lap time

Constraints:

- Avoid other cars
- Stay on road
- Don't skid
- Limited acceleration

Intuitive approach:

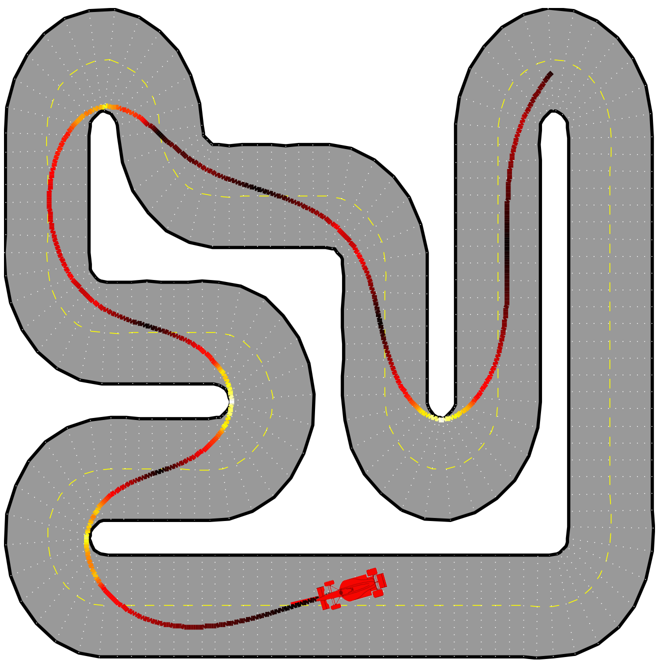
- Look forward and plan path based on
  - Road conditions
  - Upcoming corners
  - Abilities of car
  - etc...



Optimization-Based Control

Minimize	(lap time)
while	avoid other cars
	stay on road
	...

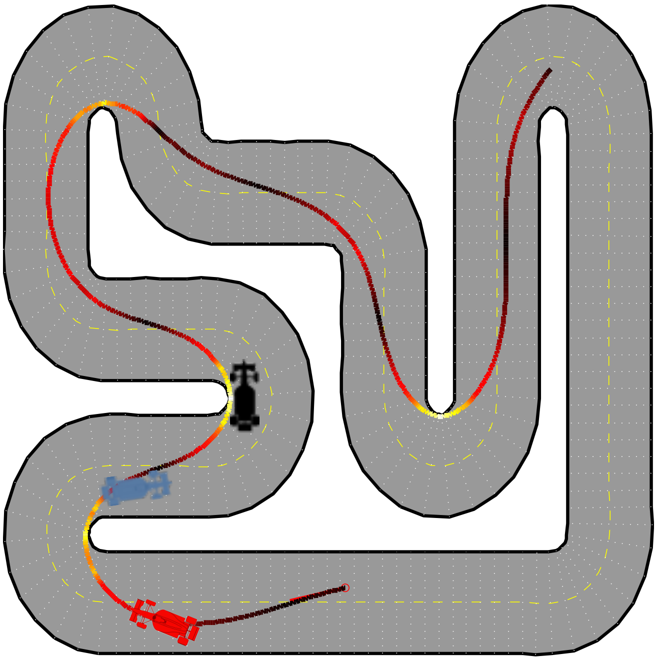
- Solve **optimization problem** to compute minimum-time path



Optimization-Based Control

Minimize	(lap time)
while	avoid other cars
	stay on road
	...

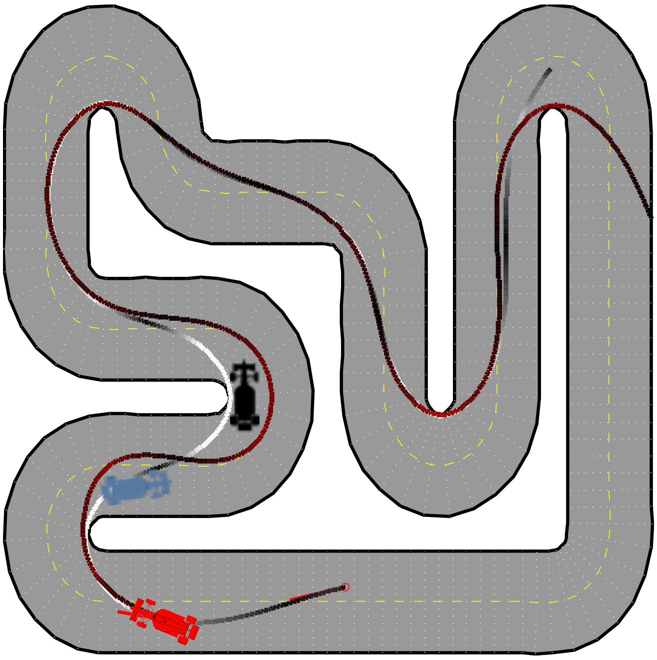
- Solve **optimization problem** to compute minimum-time path
- What to do if something unexpected happens?
  - We didn't see a car around the corner!
  - Must introduce *feedback*



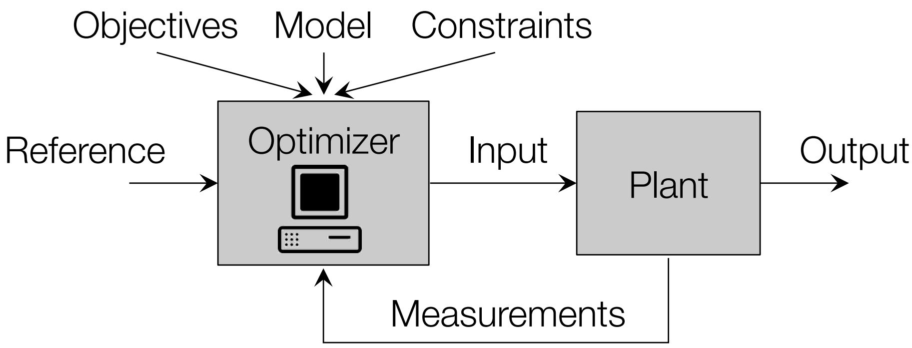
Optimization-Based Control

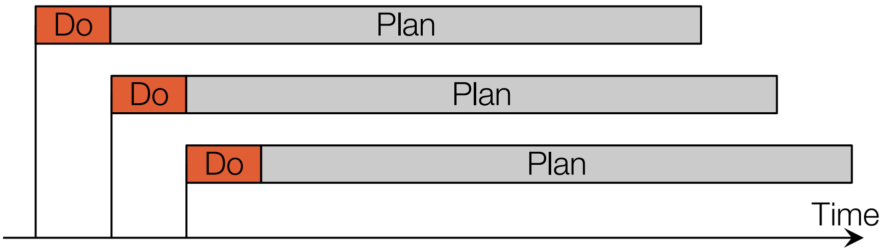
Minimize	(lap time)
while	avoid other cars
	stay on road
	...

- Solve **optimization problem** to compute minimum-time path
- Obtain series of planned control actions
- Apply *first* control action
- Repeat the planning procedure



Model Predictive Control





Receding horizon strategy introduces **feedback**.

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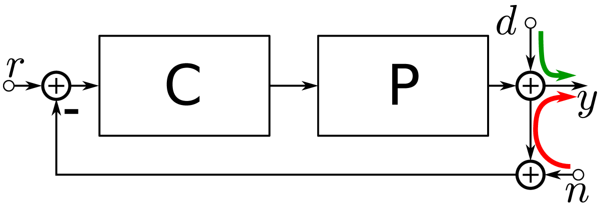
Two Different Perspectives

Classical design: design C

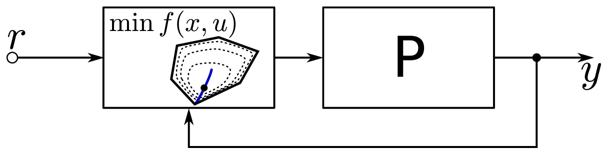


Dominant issues addressed

- Disturbance rejection $(d \to y)$
- Noise insensitivity $(n \to y)$
- Model uncertainty

(usually in *frequency domain*)

MPC: real-time, repeated optimization to choose $u(t)$



Dominant issues addressed

- Control constraints (limits)
- Process constraints (safety)

(usually in *time domain*)

Constraints in Control

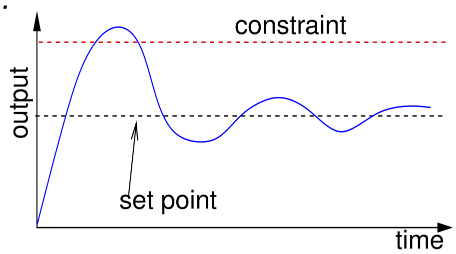
All physical systems have **constraints**:

- Physical constraints, e.g. actuator limits
- Performance constraints, e.g. overshoot
- Safety constraints, e.g. temperature/pressure limits

Optimal operating points are often near constraints.

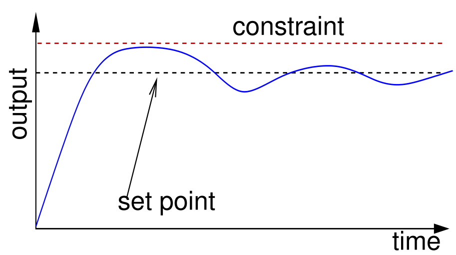
Classical control methods:

- Ad hoc constraint management
- Set point sufficiently far from constraints
- Suboptimal plant operation



Predictive control:

- Constraints included in the design
- Set point optimal
- Optimal plant operation



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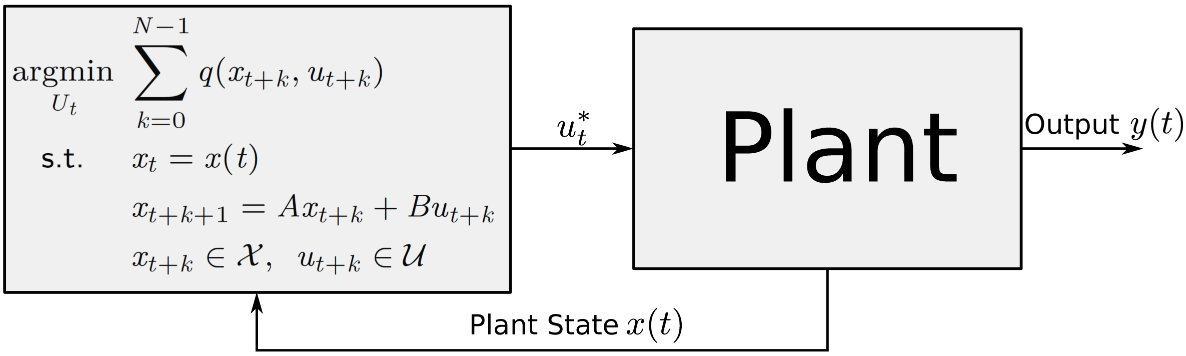
MPC: Mathematical Formulation

$$
\begin{aligned}
U_t^*(x(t)) := \underset{U_t}{\operatorname{argmin}} & \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) & \\
\text{subj. to } & x_t = x(t) & \text{measurement} \\
& x_{t+k+1} = Ax_{t+k} + Bu_{t+k} & \text{system model} \\
& x_{t+k} \in \mathcal{X} & \text{state constraints} \\
& u_{t+k} \in \mathcal{U} & \text{input constraints} \\
& U_t = \{u_t, u_{t+1}, \ldots, u_{t+N-1}\} & \text{optimization variables}
\end{aligned}
$$

Problem is defined by

- **Objective** that is minimized,
  e.g., distance from origin, sum of squared/absolute errors, economic,...
- Internal **system model** to predict system behavior
  e.g., linear, nonlinear, single-/multi-variable, ...
- **Constraints** that have to be satisfied
  e.g., on inputs, outputs, states, linear, quadratic,...

MPC: Mathematical Formulation



At each sample time:

- Measure / estimate current state $x(t)$
- Find the optimal input sequence for the entire planning window N: $U_t^* = \{u_t^*, u_{t+1}^*, \ldots, u_{t+N-1}^*\}$
- Implement only the *first* control action u_t^*

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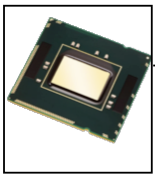
MPC: Applications



Computer control — ns

μs — Power systems

Traction control — ms

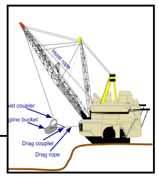
Seconds — Buildings

Refineries — Minutes

Hours — Nurse rostering

Train scheduling — Days

Weeks — Production planning



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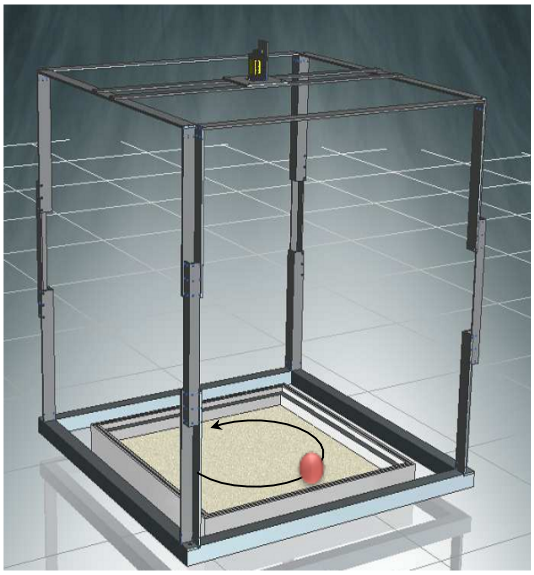
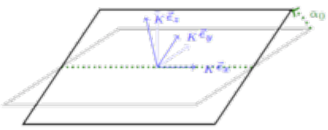
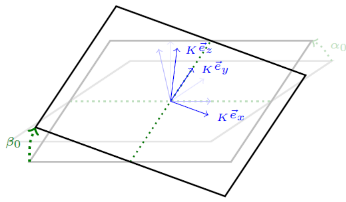
Ball on Plate

- **Movable plate** (0.66m x 0.66m)
- Can be revolved around two axis $[+17°; -17°]$ by two DC motors
- Angle is measured by potentiometers
- Position of the ball is measured by a camera
- *Model*: Linearized dynamics, 4 states, 1 input per axis
- *Input constraints*: Voltage of motors
- *State constraints*: Boundary of the plate, angle of the plate







[*R. Waldvogel. Master Thesis ETH, 2010*]

Ball on Plate

Controller comparison: LQR vs. MPC in the presence of input constraints

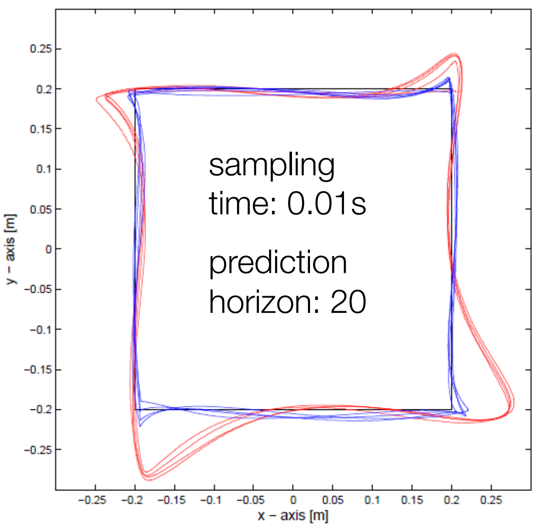


Figure : LQR (red) vs MPC (blue)

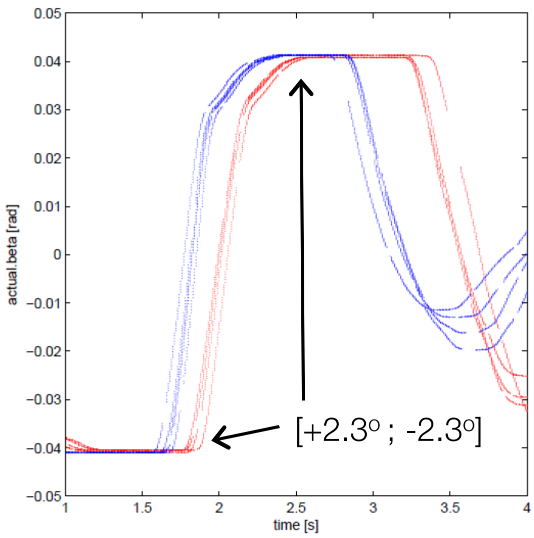


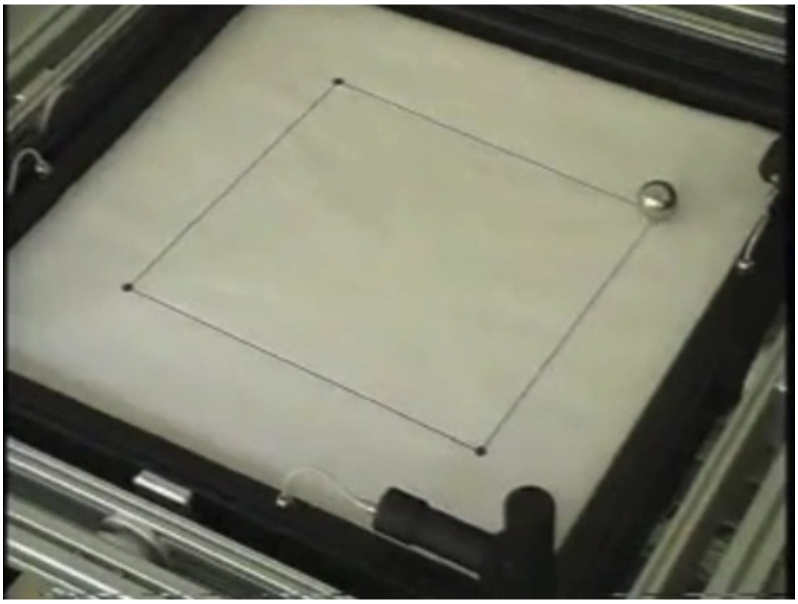
Figure : Input β for the upper left corner.

MPC introduces **preview** by predicting the state over a finite horizon

[*R. Waldvogel. Master Thesis ETH, 2010*]

Ball on Plate

MPC Control of a Ball and Plate System:



[*R. Waldvogel. Master Thesis ETH, 2010*]

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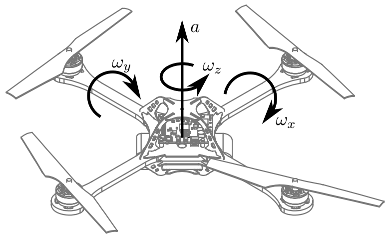
Autonomous Quadrocopter Flight

Quadrocopters:

- Highly agile due to fast rotational dynamics
- High thrust-to-weight ratio allows for large translational accelerations
- Motion control by altering rotation rate and/or pitch of the rotors
- High thrust motors enable high performance control

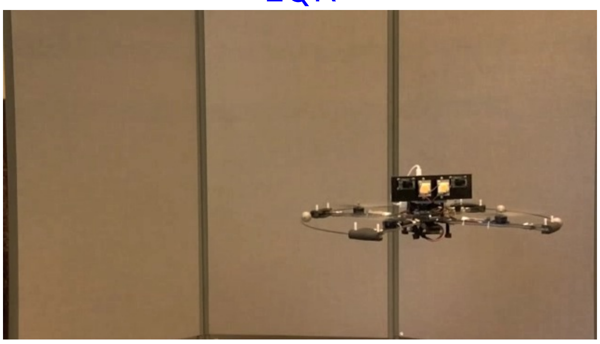


Control Problem:

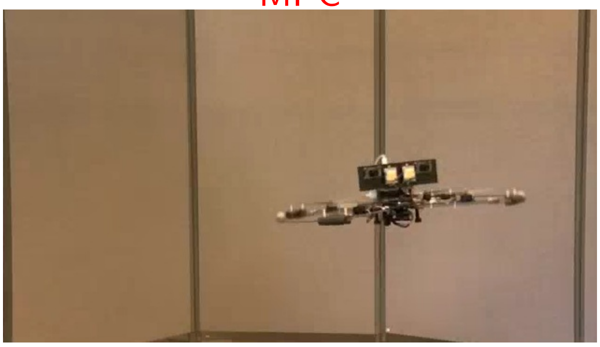
- *Nonlinear system* in 6D (position, attitude)
- *Constraints*: limited thrust, rates,...
- *Task*: Hovering, trajectory tracking
- *Challenges*: Fast unstable dynamics

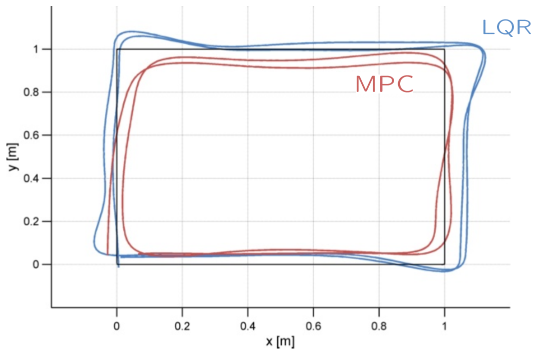
Autonomous Quadrocopter flight

LQR



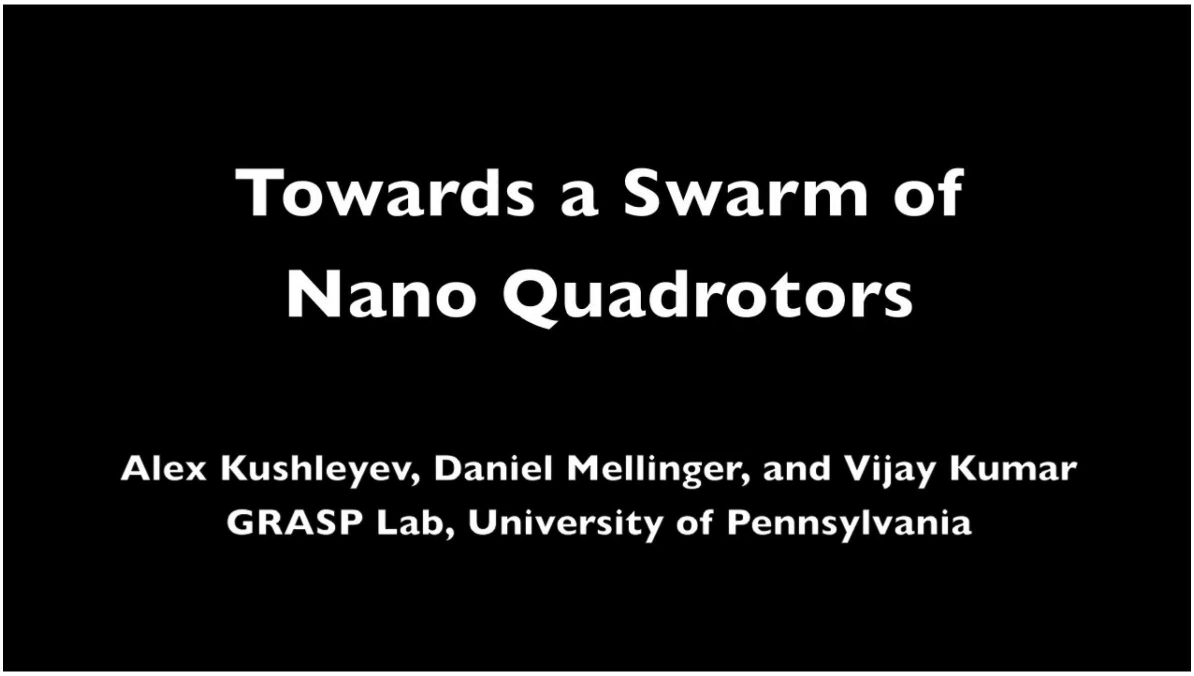
MPC





[*M. Burri. Master Thesis ETH, 2011*]

Autonomous Quadrocopter flight



[*GRASP Lab. University of Pennsylvania, 2012;* `http://www.grasp.upenn.edu/`]

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Autonomous dNaNo Race Cars

Race car:

- 1:43 scale, very light (50g) and fast
- Radio controlled
- 2.4GHz transmitter allows to run up to 40 cars

Control Problem:

- *Nonlinear model* in 4D (position, orientation)
- *Constraints:* acceleration, steering angle, race track, other cars...
- *Task:* Optimal path planning and path following
- *Challenges:* State estimation, effects that are difficult to model/measure, e.g. slip, small sampling times

Autonomous dNaNo Race Cars

[*ORCA Racer Project. ETH, 2011;* `http://orcaracer.ethz.ch/`]

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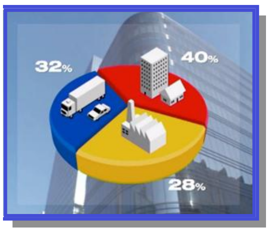
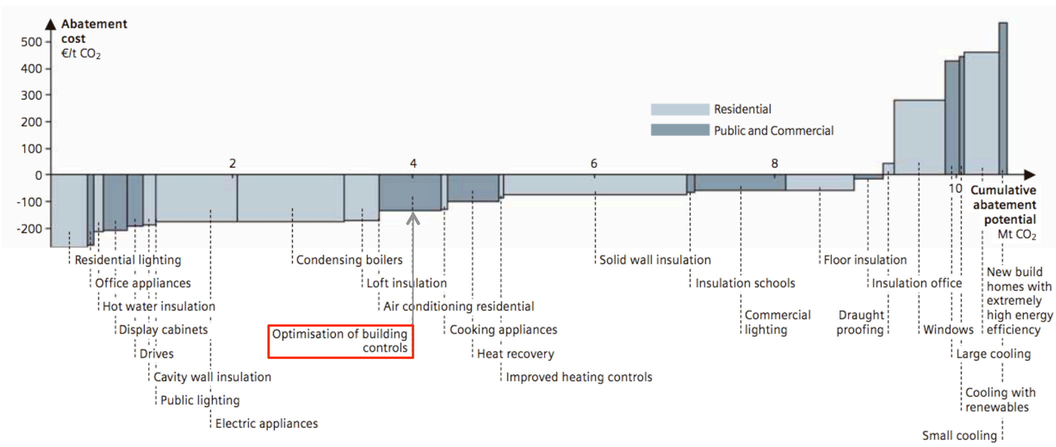
Energy Efficient Building Control

- Buildings account for approx. *40% of global energy use*
- Most energy is consumed during use of the buildings
- Building sector has large potential for cost-effective reduction of CO_2 emissions
- Most investments in buidlings are expected to pay back through *reduced energy bills*



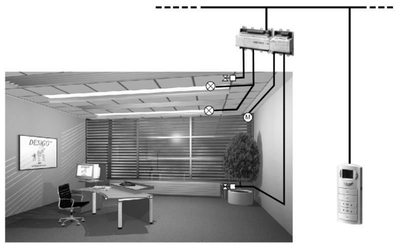


Greenhouse gas abatement cost curve for London buildings (2025, decision maker perspective)

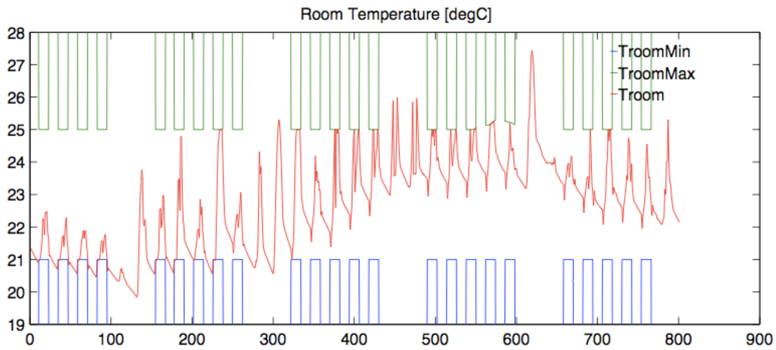
Source: Watson, J. (ed.) (2008): Sustainable Urban Infrastructure, London Edition – a view to 2025. Siemens AG, Corporate Communications (CC) Munich, 71pp.

Energy Efficient Building Control

Integrated Room Automation:
Integrated control of heating, cooling, ventilation, electrical lighting, blinds,... of a single room/zone

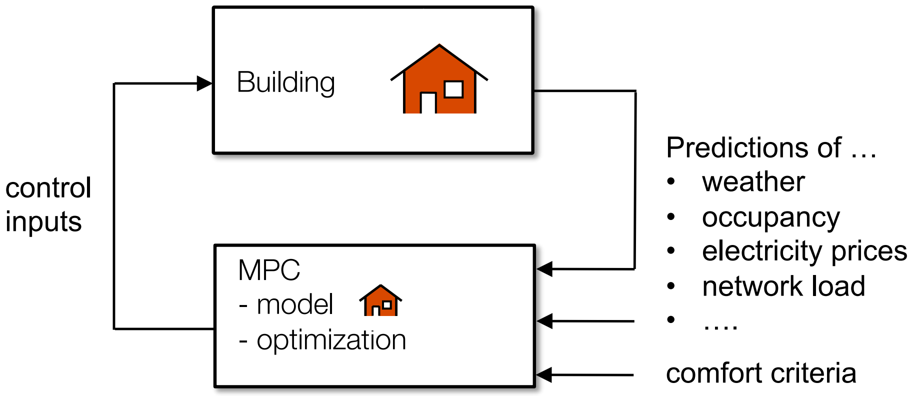


Control Task: Use minimum amount of energy (or money) to keep room temperature, illuminance level and CO_2 concentration in *prescribed comfort ranges*



[*OptiControl Project, ETH. 2010;* `http://www.opticontrol.ethz.ch/`]

Energy Efficient Building Control



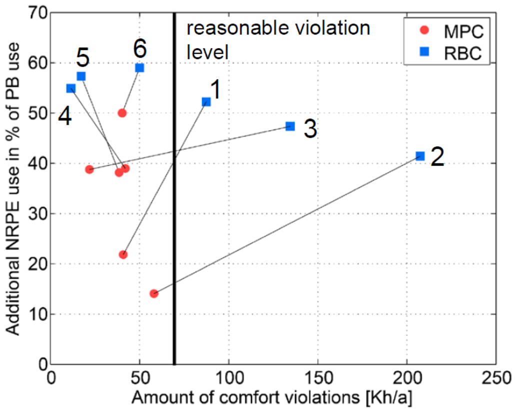
MPC opens the possibility to

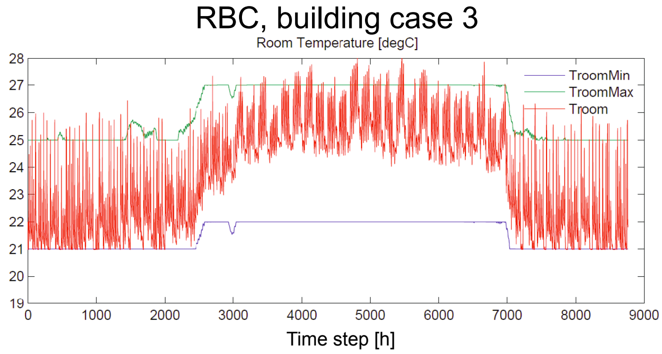
- exploit building's *thermal storage capacity*
- use *predictions* of future disturbances, e.g. weather, for better planning
- use forecasts of electricity prices to shift electricity demand for grid-friendly behavior
- offer grid-balancing services to the power network
- ...

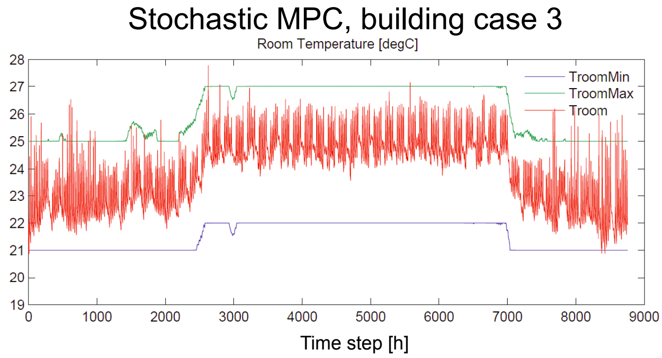
while respecting requirements for building usage (temperature, light, ...)

Energy Efficient Building Control

Optimize energy efficiency using weather predictions:







MPC: Stochastic MPC
RBC: Current best practice Rule Based Controller

[*OptiControl Project. ETH, 2010;* `http://www.opticontrol.ethz.ch/`]

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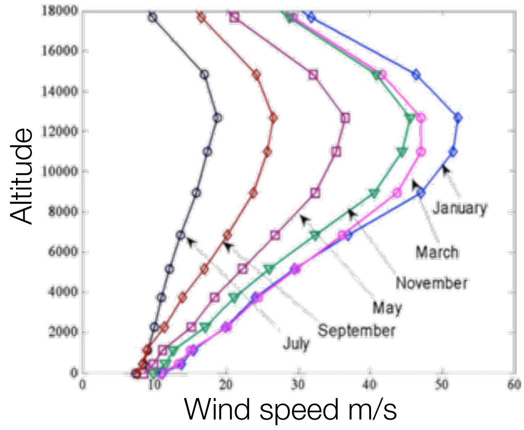
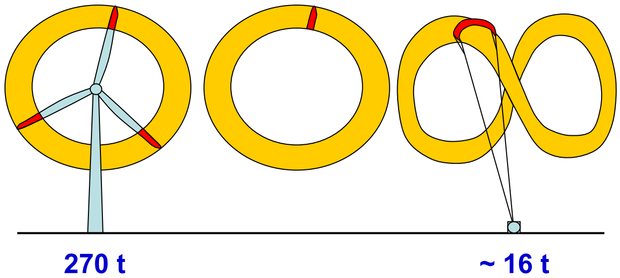
Kite Power

- Wind energy has potential to supply global energy need.
- Current wind technology is not able to exploit the potential
  - Traditional inland wind turbines are close to scaling limits
  - Economic operation only possible at a limited number of locations

Idea: Exploit the energy of high-altitude wind by means of light tethered wings (kites)
Goal: Wind power at lower cost than coal



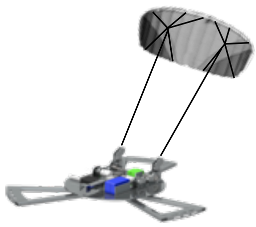


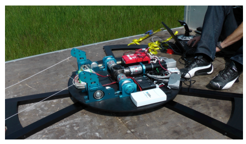
Exploit that

- Wind speed at 800m = 1.5 x speed at 80m
- Power density = (wind speed)^3

Kite Power

- Different kites proposed: flexible vs. rigid wings (different models, nonlinear)
- On board vs. ground level generator
- Ground level seems to be more viable for large-scale
- Number of lines?

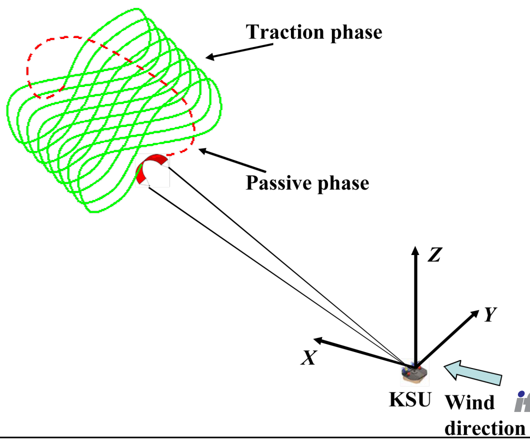




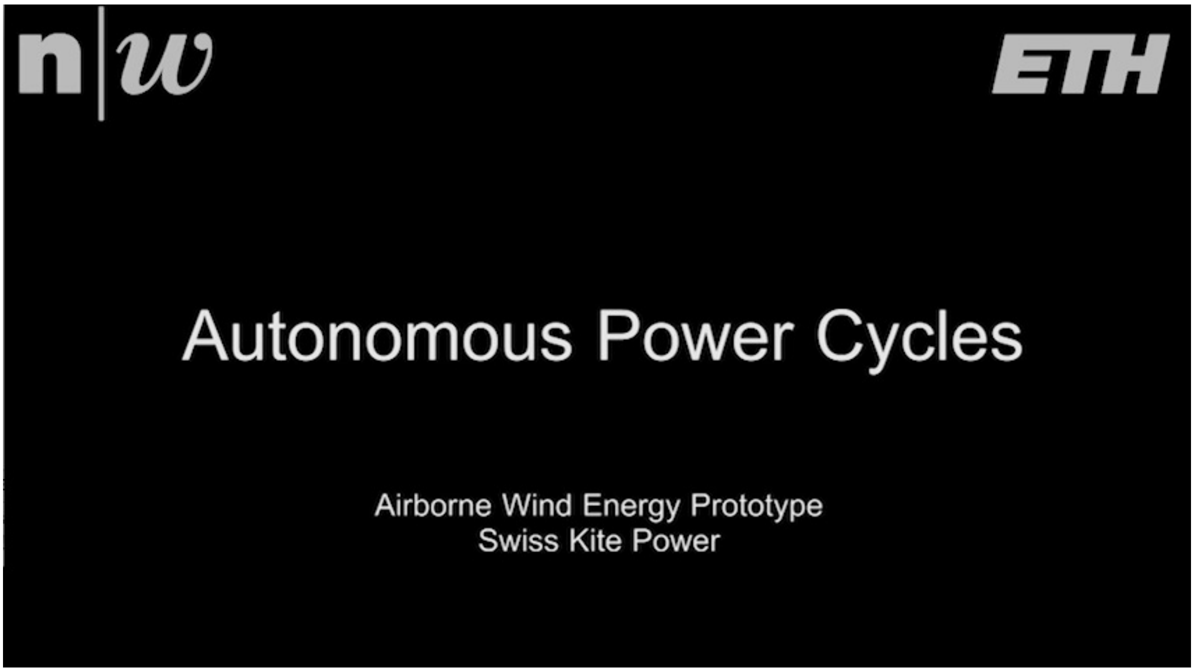
[A. Zgraggen, ETH, 2011]

Kite control problem:

- Maximize the net generated energy
- Maintain stability of the wing
- Exploit crosswind, i.e. kites fly transverse to wind at high speed
- Satisfy physical constraints: keep the kite far away from the ground, avoid line wrapping...
- Each configuration and working phase has its own performance goal



Kite Power



[*Airbone Wind Energy Group. ETH, 2013;* `http://control.ee.ethz.ch/~awe/`]

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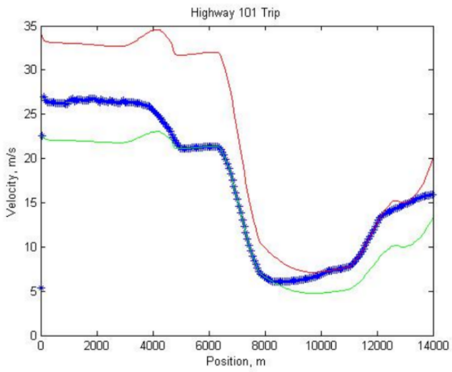
Audi Smart Engine



- **Fact:** Do not accelerate if there is a traffic jam, you will only waste fuel.
- **Idea:** Use traffic forecast to regulate the speed of a car to save fuel while getting to destination on time.

- MPC regulates the desired speed (through an Automatic Cruise Control) in order to reach the destination in the most fuel-efficient way, given a

- Min and Max traffic speed forecast and road grade used in the MPC constraints and model

Ford Autonomous Driving on Ice

- Autonomous double-lane change.
- Road forecast and nonlinear vehicle model (driving on ice) used in MPC.
- MPC controls differential braking and steering.
- Experimental results @ 72 km/h on ice.

[*Falcone, Borrelli et al. International Journal Vehicle Autonomous Systems, 2009*]

Volvo

- Autonomous lane keeping (minimally invasive).
- Road forecast and vehicle model used in MPC.
- MPC controls braking and steering.

[*Gray, Ali, Gao, Hedrick and Borrelli. IEEE Transactions on Intelligent Transportation Systems, 2013*]

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Robotic Chameleon

- Tracking an object (point in video) using two independent cameras.
- MPC controls cameras pan tilt and zoom to keep object in a given field of view (constraints).
- MPC uses cameras models and forecast the object position (assuming moving at constant acceleration over the prediction horizon).
- Experimental results with MPC solved at 100 Hz.

[*Avin, Borrelli et al. Autonomous Robots, 2008*]

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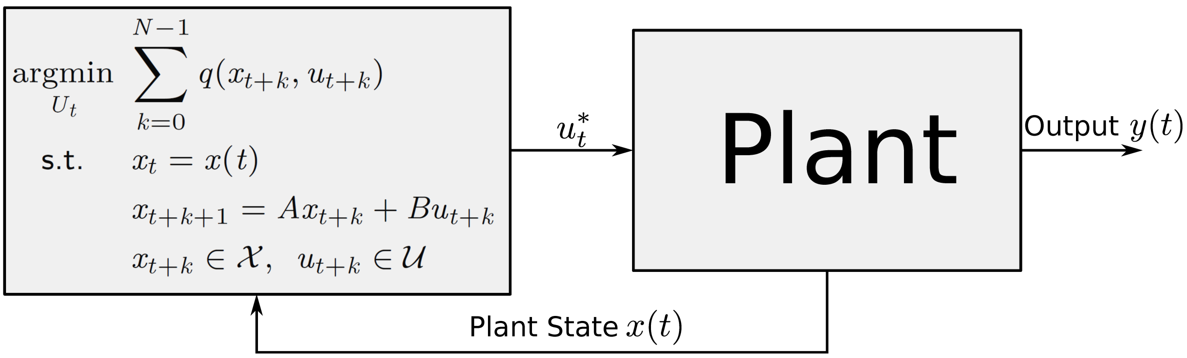
Summary: MPC



At each sample time:

- Measure /estimate current state $x(t)$
- Find the *optimal input sequence* for the entire planning window N: $U_t^* = \{u_t^*, u_{t+1}^*, \ldots, u_{t+N-1}^*\}$
- Implement only the *first* control action u_t^*

Summary

- Obtain a model of the system
- Design a state observer
- Define optimal control problem
- Set up optimization problem in optimization software
- Solve optimization problem to get optimal control sequence
- Verify that closed-loop system performs as desired,
  e.g., check performance criteria, robustness, real-time aspects,...

Important Aspects of Model Predictive Control

Main advantages:

- Systematic approach for handling *constraints*
- High *performance* controller

Main challenges:

- *Implementation*
  MPC problem has to be solved in real-time, i.e. within the sampling interval of the system, and with available hardware (storage, processor,...).
- *Stability*
  Closed-loop stability, i.e. convergence, is not automatically guaranteed
- *Robustness*
  The closed-loop system is not necessarily robust against uncertainties or disturbances
- *Feasibility*
  Optimization problem may become infeasible at some future time step, i.e. there may not exist a plan satisfying all constraints

Outlook

- Part II: Constrained Finite Time Optimal Control
  Formulating and solving the optimization problem online
- Part III: Feasibility and Stability
  Guaranteeing feasibility and stability by design
- Advanced Topics
  Tracking, Soft-Constraints, Explicit MPC, Hybrid Systems

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Literature

Model Predictive Control:

- Predictive Control for linear and hybrid systems, F. Borrelli, A. Bemporad, M. Morari, 2013 Cambridge University Press [`http://www.mpc.berkeley.edu/mpc-course-material`]
- Model Predictive Control: Theory and Design, James B. Rawlings and David Q. Mayne, 2009 Nob Hill Publishing
- Predictive Control with Constraints, Jan Maciejowski, 2000 Prentice Hall

Optimization:

- Convex Optimization, Stephen Boyd and Lieven Vandenberghe, 2004 Cambridge University Press
- Numerical Optimization, Jorge Nocedal and Stephen Wright, 2006 Springer

Model Predictive Control
Part II – Constrained Finite Time Optimal Control

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Spring Semester 2015

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Constrained Linear Optimal Control

Cost function

$$J_0(x(0), U_0) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$$

- $U_0 \triangleq [u_0', \dots, u_{N-1}']'$
- Squared Euclidian norm: $p(x_N) = x_N' P x_N$ and $q(x_k, u_k) = x_k' Q x_k + u_k' R u_k$.
- $p = 1$ or $p = \infty$: $p(x_N) = \|Px_N\|_p$ and $q(x_k, u_k) = \|Qx_k\|_p + \|Ru_k\|_p$.

Constrained Finite Time Optimal Control problem (CFTOC)

$$\begin{array}{rll} J_0^*(x(0)) = & \min_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \dots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \dots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \qquad (1)$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

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Feasible Sets

Set of initial states $x(0)$ for which the optimal control problem (1) is feasible:

$$\begin{array}{rl}\mathcal{X}_0 = & \{x_0 \in \mathbb{R}^n | \ \exists (u_0, \ldots, u_{N-1}) \text{ such that } x_k \in \mathcal{X}, \ u_k \in \mathcal{U}, \\ & k = 0, \ldots, N-1, \ x_N \in \mathcal{X}_f, \ \text{where } x_{k+1} = Ax_k + Bu_k\}\end{array}$$

In general $\mathcal{X}_i$ is the set of states x_i at time i for which (1) is feasible:

$$\begin{array}{rl}\mathcal{X}_i = & \{x_i \in \mathbb{R}^n | \ \exists (u_i, \ldots, u_{N-1}) \text{ such that } x_k \in \mathcal{X}, \ u_k \in \mathcal{U}, \\ & k = i, \ldots, N-1, \ x_N \in \mathcal{X}_f, \ \text{where } x_{k+1} = Ax_k + Bu_k\},\end{array}$$

The sets $\mathcal{X}_i$ for $i = 0, \ldots, N$ play an important role in the the solution of the CFTOC problem. They are independent of the cost.

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Unconstrained Solution
Results from Lectures on Days 1 & 2

For quadratic cost (squared Euclidian norm) and **no state and input constraints**:

$$\{x \in \mathcal{X},\ u \in \mathcal{U}\} = \mathbb{R}^{n+m},\ \mathcal{X}_f = \mathbb{R}^n$$

we have the *time-varying* linear control law

$$u^*(k) = F_k x(k)\ \ k = 0, \dots, N-1.$$

If $N \to \infty$, we have the *time-invariant* linear control law

$$u^*(k) = F_\infty x(k)\ \ k = 0, 1, \dots$$

Next we show how to compute finite time **constrained** optimal controllers.

Outline

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Problem Formulation

Quadratic cost function

$$J_0(x(0), U_0) = x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \tag{2}$$

with $P \succeq 0,\ Q \succeq 0,\ R \succ 0$.
Constrained Finite Time Optimal Control problem (CFTOC).

$$\begin{array}{lll} J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{3}$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

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Construction of the QP with substitution

- **Step 1**: Rewrite the cost as (see lectures on Day 1 & 2)

$$\begin{aligned} J_0(x(0), U_0) &= U_0' H U_0 + 2x(0)' F U_0 + x(0)' Y x(0) \\ &= [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \end{aligned}$$

  Note: $\left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] \succeq 0$ since $J_0(x(0), U_0) \geq 0$ by assumption.

- **Step 2**: Rewrite the constraints compactly as (details provided on the next slide)

$$G_0 U_0 \leq w_0 + E_0 x(0)$$

- **Step 3**: Rewrite the optimal control problem as

$$\begin{aligned} J_0^*(x(0)) = \min_{U_0} \quad & [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \\ \text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0) \end{aligned}$$

Solution

$$\begin{aligned} J_0^*(x(0)) = \min_{U_0} \quad & [U_0' \ x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \ x(0)']' \\ \text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0) \end{aligned}$$

For a given x(0) U_0^* can be found via a QP solver.

Construction of QP constraints with substitution

If $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_f$ are given by:

$$\mathcal{X} = \{x \,|\, A_x x \leq b_x\} \qquad \mathcal{U} = \{u \,|\, A_u u \leq b_u\} \qquad \mathcal{X}_f = \{x \,|\, A_f x \leq b_f\}$$

Then G_0, E_0 and w_0 are defined as follows

$$G_0 = \begin{bmatrix} A_u & 0 & \dots & 0 \\ 0 & A_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_u \\ 0 & 0 & \dots & 0 \\ A_x B & 0 & \dots & 0 \\ A_x A B & A_x B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_f A^{N-1} B & A_f A^{N-2} B & \dots & A_f B \end{bmatrix}, E_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix}, w_0 = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

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Construction of the QP without substitution

To obtain the QP problem

$$\begin{aligned} J_0^*(x(0)) = \min_{U_0} \quad & [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Q \end{smallmatrix}\right] [U_0{}' \; x(0)']' \\ \text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0) \end{aligned}$$

we have substituted the state equations

$$x_{k+1} = Ax_k + Bu_k$$

into the state constraints $x_k \in \mathcal{X}$.

It is often more efficient to keep the explicit equality constraints.

Construction of the QP without substitution

We transform the CFTOC problem into the QP problem

$$
\begin{aligned}
J_0^*(x(0)) = \min_z \quad & [z' \ x(0)'] \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} [z' \ x(0)']' \\
\text{subj. to } \ & G_{0,\text{in}} z \le w_{0,\text{in}} + E_{0,\text{in}} x(0) \\
& G_{0,\text{eq}} z = E_{0,\text{eq}} x(0)
\end{aligned}
$$

- Define variable:

$$
z = \begin{bmatrix} x_1' & \ldots & x_N' & u_0' & \ldots & u_{N-1}' \end{bmatrix}'
$$

- Equalities from system dynamics $x_{k+1} = Ax_k + Bu_k$:

$$
G_{0,\text{eq}} = \left[\begin{array}{ccccc:ccccc}
I & & & & & -B & & & & \\
-A & I & & & & & -B & & & \\
& -A & I & & & & & -B & & \\
& & \ddots & \ddots & & & & & \ddots & \\
& & & -A & I & & & & & -B
\end{array}\right], \quad E_{0,\text{eq}} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Construction of the QP without substitution

If $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_f$ are given by:

$$
\mathcal{X} = \{x \,|\, A_x x \le b_x\} \qquad \mathcal{U} = \{u \,|\, A_u u \le b_u\} \qquad \mathcal{X}_f = \{x \,|\, A_f x \le b_f\}
$$

Then matrices $G_{0,\text{in}}$, $w_{0,\text{in}}$ and $E_{0,\text{in}}$ are:

$$
G_{0,\text{in}} = \left[\begin{array}{ccccc:ccccc}
0 & & & & & 0 & & & & \\
& A_x & & & & & 0 & & & \\
& & \ddots & & & & & \ddots & & \\
& & & A_x & & & & & 0 & \\
& & & & A_f & & & & & 0 \\ \hdashline
0 & & & & & A_u & & & & \\
& 0 & & & & & A_u & & & \\
& & \ddots & & & & & \ddots & & \\
& & & 0 & & & & & A_u & \\
& & & & 0 & & & & & A_u
\end{array}\right] \qquad
w_{0,\text{in}} = \left[\begin{array}{c}
b_x \\ b_x \\ \vdots \\ b_x \\ b_f \\ \hdashline b_u \\ b_u \\ \vdots \\ b_u \\ b_u
\end{array}\right]
$$

$$
E_{0,\text{in}} = \begin{bmatrix} -A_x' & 0 & \cdots & 0 \end{bmatrix}'
$$

Construction of the QP without substitution

Build cost function from MPC cost $x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$

$$\bar{H} = \left[\begin{array}{cccc|ccc} Q & & & & & & \\ & \ddots & & & & & \\ & & Q & & & & \\ & & & P & & & \\ \hline & & & & R & & \\ & & & & & \ddots & \\ & & & & & & R \end{array}\right]$$

Matlab hint:

```
barH = blkdiag(kron(eye(N-1),Q), P, kron(eye(N),R))
```


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2-Norm State Feedback Solution

Start from QP with substitution.

- **Step 1**: Define $z \triangleq U_0 + H^{-1}F'x(0)$ and transform the problem into

$$\hat{J}^*(x(0)) = \min_z \quad z'Hz \qquad \text{subj. to} \quad G_0 z \leq w_0 + S_0 x(0),$$

  where $S_0 \triangleq E_0 + G_0 H^{-1}F'$, and
  $\hat{J}^*(x(0)) = J_0^*(x(0)) - x(0)'(Y - FH^{-1}F')x(0)$.

  The CFTOC problem is now a **multiparametric quadratic program (mp-QP)**.

- **Step 2**: Solve the mp-QP to get explicit solution $z^*(x(0))$
- **Step 3**: Obtain $U_0^*(x(0))$ from $z^*(x(0))$

2-Norm State Feedback Solution

Main Results

1. The **Open loop optimal control function** can be obtained by solving the mp-QP problem and calculating $U_0^*(x(0)),\ \forall x(0) \in \mathcal{X}_0$ as $U_0{}^* = z^*(x(0)) - H^{-1}F'x(0)$.
2. The first component of the multiparametric solution has the form

$$u^*(0) = f_0(x(0)), \quad \forall x(0) \in \mathcal{X}_0,$$

   $f_0 : \mathbb{R}^n \to \mathbb{R}^m$, is continuous and PieceWise Affine on Polyhedra

$$f_0(x) = F_0^i x + g_0^i \quad \text{if} \quad x \in CR_0^i, \quad i = 1, \ldots, N_0^r$$

3. The polyhedral sets $CR_0^i = \{x \in \mathbb{R}^n | H_0^i x \leq K_0^i\}$, $i = 1, \ldots, N_0^r$ are a partition of the feasible polyhedron $\mathcal{X}_0$.
4. The value function $J_0^*(x(0))$ is convex and piecewise quadratic on polyhedra.

Example

Consider the double integrator

$$\begin{cases} x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to constraints

$$-1 \le u(k) \le 1, \ k = 0, \dots, 5$$

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x(k) \le \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \ k = 0, \dots, 5$$

Compute the **state feedback** optimal controller $u^*(0)(x(0))$ solving the CFTOC

problem with $N = 6$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 0.1$, P the solution of the ARE, $\mathcal{X}_f = \mathbb{R}^2$.

Example

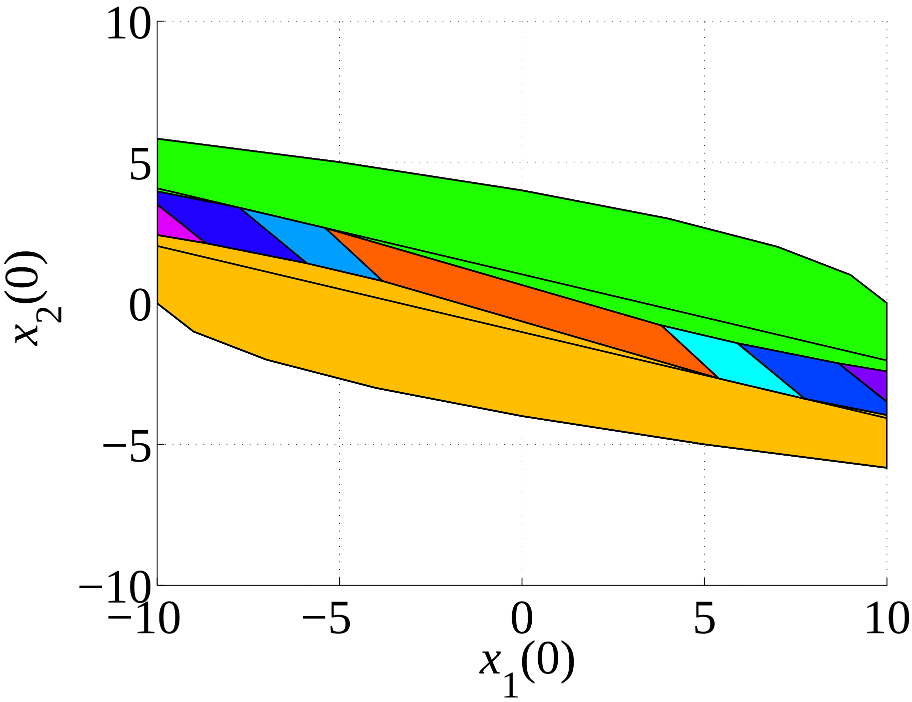

Figure : Partition of the state space for the affine control law $u^*(0)$ ($N_0^r = 13$)

Outline

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Problem Formulation

Piece-wise linear cost function

$$J_0(x(0), U_0) := \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p \tag{4}$$

with $p = 1$ or $p = \infty$, P, Q, R full column rank matrices

Constrained Finite Time Optimal Control Problem (CFTOC)

$$\begin{array}{rll} J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k, \;\; k = 0, \dots, N-1 \\ & & x_k \in \mathcal{X}, \;\; u_k \in \mathcal{U}, \;\; k = 0, \dots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{5}$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

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Construction of the LP with substitution

Recall that the $\infty-$norm problem can be equivalently formulated as

$$
\begin{aligned}
\min_{z_0} \quad & \varepsilon_0^x + \ldots + \varepsilon_N^x + \varepsilon_0^u + \ldots + \varepsilon_{N-1}^u \\
\text{subj. to} \quad & -\mathbf{1}_n \varepsilon_k^x \leq \pm Q \left[A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \right], \\
& -\mathbf{1}_r \varepsilon_N^x \leq \pm P \left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \right], \\
& -\mathbf{1}_m \varepsilon_k^u \leq \pm R u_k, \\
& A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \in \mathcal{X}, \; u_k \in \mathcal{U}, \\
& A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \in \mathcal{X}_f, \\
& k = 0, \ldots, N-1 \\
& x_0 = x(0)
\end{aligned}
$$

Construction of the LP with substitution

The problem results in the following standard LP

$$
\begin{aligned}
\min_{z_0} \quad & c_0' z_0 \\
\text{subj. to} \quad & \bar{G}_0 z_0 \leq \bar{w}_0 + \bar{S}_0 x(0)
\end{aligned}
$$

where $z_0 := \{\varepsilon_0^x, \ldots, \varepsilon_N^x, \varepsilon_0^u, \ldots, \varepsilon_{N-1}^u, u_0', \ldots, u_{N-1}'\} \in \mathbb{R}^s$, $s \triangleq (m+1)N + N + 1$ and

$$
\bar{G}_0 = \begin{bmatrix} G_\varepsilon & 0 \\ 0 & G_0 \end{bmatrix}, \; \bar{S}_0 = \begin{bmatrix} S_\varepsilon \\ S_0 \end{bmatrix}, \; \bar{w}_0 = \begin{bmatrix} w_\varepsilon \\ w_0 \end{bmatrix}
$$

For a given $x(0)$ U_0^* can be obtained via an LP solver (the $1-$norm case is similar).

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1- /∞-Norm State Feedback Solution

Main Results

1. The **Open loop optimal control function** can be obtained by solving the mp-LP problem and calculating $z_0^*(x(0))$
2. The component $u_0^* = [0 \ \ldots 0 \ I_m \ 0 \ \ldots \ 0] z_0^*(x(0))$ of the multiparametric solution has the form

$$u^*(0) = f_0(x(0)), \quad \forall x(0) \in \mathcal{X}_0,$$

$f_0 : \mathbb{R}^n \to \mathbb{R}^m$, is continuous and PieceWise Affine on Polyhedra

$$f_0(x) = F_0^i x + g_0^i \quad \text{if} \quad x \in CR_0^i, \quad i = 1, \ldots, N_0^r$$

3. The polyhedral sets $CR_0^i = \{x \in \mathbb{R}^n | H_0^i x \leq K_0^i\}$, $i = 1, \ldots, N_0^r$ are a partition of the feasible polyhedron $\mathcal{X}_0$.
4. In case of multiple optimizers a PieceWise Affine control law exists.
5. The value function $J_0^*(x(0))$ is convex and piecewise linear on polyhedra.

Model Predictive Control
Part III – Feasibility and Stability

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Spring Semester 2015

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Infinite Time Constrained Optimal Control (what we would like to solve)

$$
\begin{aligned}
J_0^*(x(0)) = \min \sum_{k=0}^{\infty} q(x_k, u_k) \\
\text{s.t. } & x_{k+1} = Ax_k + Bu_k, k = 0, \ldots, N-1 \\
& x_k \in \mathcal{X}, u_k \in \mathcal{U}, k = 0, \ldots, N-1 \\
& x_0 = x(0)
\end{aligned}
$$

- **Stage cost** $q(x, u)$ describes "cost" of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties...
  ... but we can't compute it: there are an **infinite number of variables**

Receding Horizon Control (what we can sometimes solve)

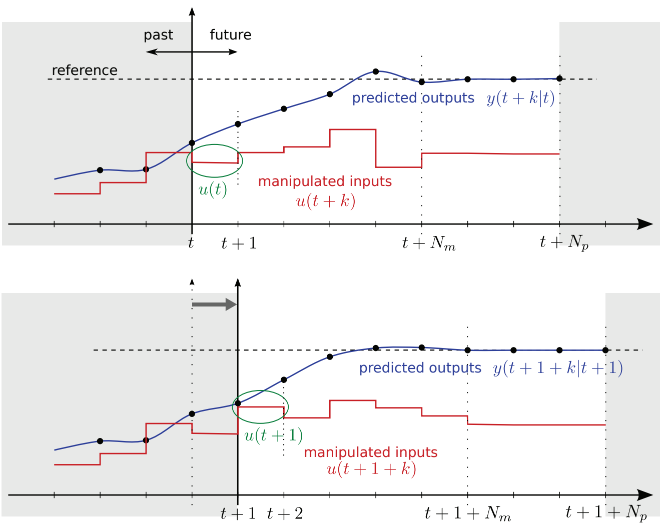
$$
\begin{aligned}
J_t^*(x(t)) = \quad & \min_{U_t} \quad p(x_{t+N}) + \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) \\
& \text{subj. to} \quad x_{t+k+1} = Ax_{t+k} + Bu_{t+k}, \; k = 0, \dots, N-1 \\
& \qquad\qquad x_{t+k} \in \mathcal{X}, \; u_{t+k} \in \mathcal{U}, \; k = 0, \dots, N-1 \\
& \qquad\qquad x_{t+N} \in \mathcal{X}_f \\
& \qquad\qquad x_t = x(t)
\end{aligned}
\tag{1}
$$

where $\mathcal{U}_t = \{u_t, \dots, u_{t+N-1}\}$.

Truncate after a finite horizon:

- $p(x_{t+N})$: Approximates the 'tail' of the cost
- $\mathcal{X}_f$: Approximates the 'tail' of the constraints

On-line Receding Horizon Control



1. At each sampling time, solve a **CFTOC**.
2. Apply the optimal input **only during** $[t, t+1]$
3. At $t+1$ solve a CFTOC over a **shifted horizon** based on new state measurements
4. The resultant controller is referred to as **Receding Horizon Controller (RHC)** or **Model Predictive Controller (MPC)**.

On-line Receding Horizon Control

1) MEASURE the state $x(t)$ at time instance t
2) OBTAIN $U_t^*(x(t))$ by solving the optimization problem in (1)
3) IF $U_t^*(x(t)) = \emptyset$ THEN 'problem infeasible' STOP
4) APPLY the first element u_t^* of U_t^* to the system
5) WAIT for the new sampling time $t+1$, GOTO 1)

Note that, we need a constrained optimization solver for step 2).

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History of MPC

- **A. I. Propoi, 1963**, "Use of linear programming methods for synthesizing sampled-data automatic systems", *Automation and Remote Control*.
- **J. Richalet et al., 1978** "Model predictive heuristic control- application to industrial processes". *Automatica*, 14:413-428.
  - known as **IDCOM (Identification and Command)**
  - impulse response model for the plant, linear in inputs or internal variables (**only stable plants**)
  - quadratic performance objective over a finite prediction horizon
  - future plant output behavior specified by a reference trajectory
  - **ad hoc** input and output constraints
  - optimal inputs computed using a heuristic iterative algorithm, interpreted as the dual of identification
  - controller was not a transfer function, hence called **heuristic**

History of MPC

- 1970s: Cutler suggested MPC in his PhD proposal at the University of Houston in 1969 and introduced it later at Shell under the name Dynamic Matrix Control. **C. R. Cutler, B. L. Ramaker, 1979** "Dynamic matrix control – a computer control algorithm". *AICHE National Meeting*, Houston, TX.
  - successful in the petro-chemical industry
  - linear step response model for the plant
  - quadratic performance objective over a finite prediction horizon
  - future plant output behavior specified by trying to follow the set-point as closely as possible
  - input and output constraints included in the formulation
  - optimal inputs computed as the solution to a least–squares problem
  - **ad hoc** input and output constraints. Additional equation added online to account for constraints. Hence a **dynamic matrix** in the least squares problem.
- **C. Cutler, A. Morshedi, J. Haydel, 1983**. "An industrial perspective on advanced control". *AICHE Annual Meeting*, Washington, DC.
  - Standard QP problem formulated in order to systematically account for constraints.

History of MPC

- Mid 1990s: extensive theoretical effort devoted to provide conditions for guaranteeing feasibility and closed-loop stability
- 2000s: development of tractable robust MPC approaches; nonlinear and hybrid MPC; MPC for very fast systems
- 2010s: stochastic MPC; distributed large-scale MPC; economic MPC

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RHC Notation

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{aligned}
$$

$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U},\ \forall t \geq 0$$

The CFTOC Problem

$$
\begin{aligned}
J_t^*(x(t)) = \quad & \min_{U_{t\to t+N|t}} \quad p(x_{t+N|t}) + \sum_{k=0}^{N-1} q(x_{t+k|t}, u_{t+k|t}) \\
& \text{subj. to} \quad x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t},\ k = 0, \ldots, N-1 \\
& \qquad\qquad x_{t+k|t} \in \mathcal{X},\ u_{t+k|t} \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& \qquad\qquad x_{t+N|t} \in \mathcal{X}_f \\
& \qquad\qquad x_{t|t} = x(t)
\end{aligned}
$$

with $U_{t\to t+N|t} = \{u_{t|t}, \ldots, u_{t+N-1|t}\}$.

RHC Notation

- $x(t)$ is the state of the system at time t.
- $x_{t+k|t}$ is the state of the model at time $t+k$, predicted at time t obtained by starting from the current state $x_{t|t} = x(t)$ and applying to the system model

$$x_{t+1|t} = Ax_{t|t} + Bu_{t|t}$$

  the input sequence $u_{t|t}, \ldots, u_{t+k-1|t}$.
- For instance, $x_{3|1}$ represents the predicted state at time 3 when the prediction is done at time $t = 1$ starting from the current state $x(1)$. It is different, in general, from $x_{3|2}$ which is the predicted state at time 3 when the prediction is done at time $t = 2$ starting from the current state $x(2)$.
- Similarly $u_{t+k|t}$ is read as "the input u at time $t+k$ computed at time t".

RHC Notation

- Let $U^*_{t\to t+N|t} = \{u^*_{t|t}, \ldots, u^*_{t+N-1|t}\}$ be the optimal solution. The first element of $U^*_{t\to t+N|t}$ is applied to system

$$u(t) = u^*_{t|t}(x(t)).$$

- The CFTOC problem is reformulated and solved at time $t+1$, based on the new state $x_{t+1|t+1} = x(t+1)$.

Receding horizon control law

$$f_t(x(t)) = u^*_{t|t}(x(t))$$

Closed loop system

$$x(t+1) = Ax(t) + Bf_t(x(t)) \triangleq f_{cl}(x(t)),\ \ t \geq 0$$

RHC Notation: Time-invariant Systems

As the system, the constraints and the cost function are time-invariant, the solution $f_t(x(t))$ becomes a time-invariant function of the initial state $x(t)$. Thus, we can simplify the notation as

$$\begin{array}{lll} J_0^*(x(t)) = & \min\limits_{U_0} & p(x_N) + \sum\limits_{k=0}^{N-1} q(x_k, u_k) \\ & \text{subj. to} & \\ & & x_{k+1} = Ax_k + Bu_k,\ \ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ \ u_k \in \mathcal{U},\ \ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(t) \end{array}$$

where $U_0 = \{u_0, \ldots, u_{N-1}\}$.

The control law and closed loop system are **time-invariant** as well, and we write $f_0(x_0)$ for $f_t(x(t))$.

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MPC Features

Pros

- Any model
  - linear
  - nonlinear
  - single/multivariable
  - time delays
  - constraints
- Any objective:
  - sum of squared errors
  - sum of absolute errors (i.e., integral)
  - worst error over time
  - economic objective

Cons

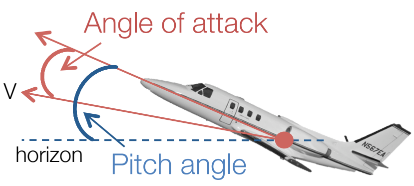
- Computationally demanding in the general case
- May or may not be stable
- May or may not be feasible

Example: Cessna Citation Aircraft

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad ($\pm 60°$), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: 0, 0, $-1.5594 \pm 2.29i$)

LQR and Linear MPC with Quadratic Cost

- Quadratic cost
- Linear system dynamics
- Linear constraints on inputs and states

LQR

$$J_\infty(x(t)) = \min \sum_{k=0}^{\infty} x_t^T Q x_t + u_k^T R u_k$$
$$\text{s.t. } x_{k+1} = Ax_k + Bu_k$$
$$x_0 = x(t)$$

Assume: $Q = Q^T \succeq 0$, $R = R^T \succ 0$

MPC

$$J_0^*(x(t)) = \min_{U_0} \sum_{k=0}^{N-1} x_k{}^T Q x_k + u_k{}^T R u_k$$
$$\text{s.t. } x_{k+1} = Ax_k + Bu_k$$
$$x_k \in \mathcal{X},\ u_k \in \mathcal{U}$$
$$x_0 = x(t)$$

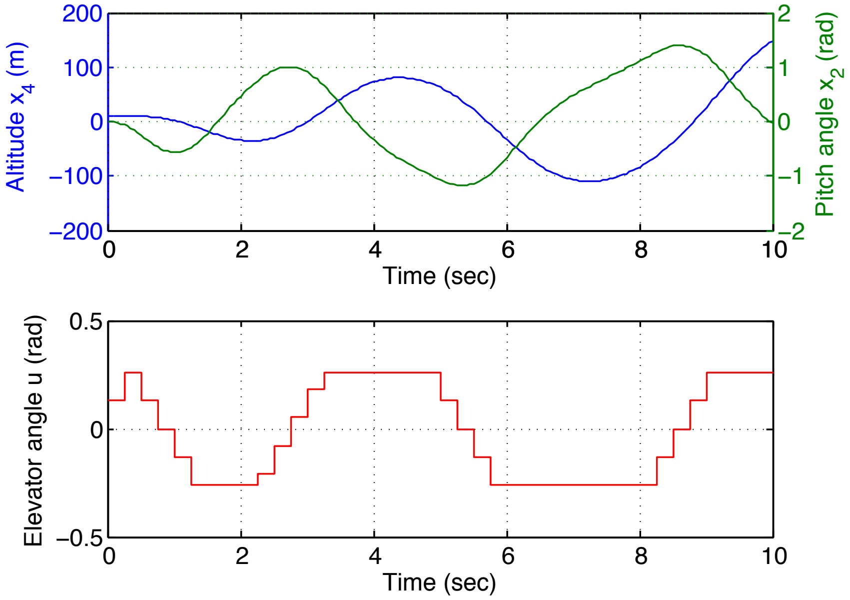
Example: LQR with saturation

Linear quadratic regulator with saturated inputs.

At time $t = 0$ the plane is flying with a deviation of $10m$ of the desired altitude, i.e. $x_0 = [0; 0; 0; 10]$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10$



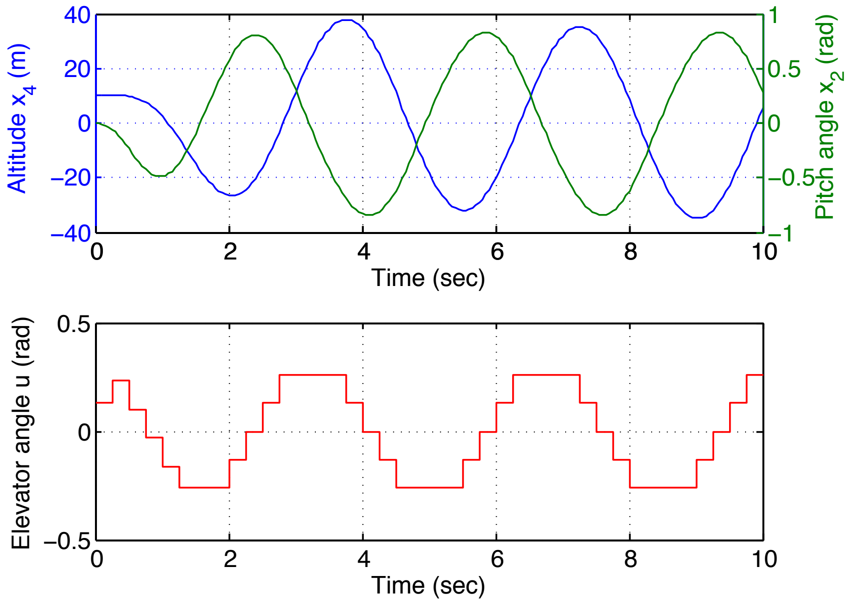
- Closed-loop system is unstable
- Applying LQR control and saturating the controller can lead to instability!

Example: MPC with Bound Constraints on Inputs

MPC controller with input constraints $|u_i| \leq 0.262$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 10$



The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

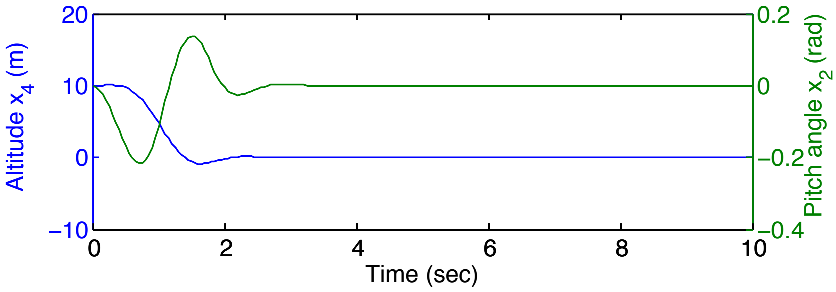
$\Rightarrow$ System does not converge to desired steady-state but to a limit cycle

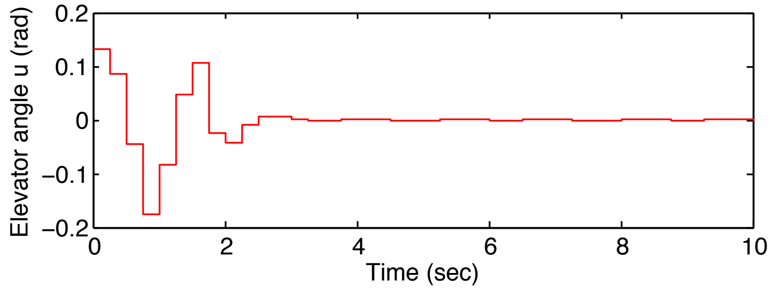
Example: MPC with all Input Constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 10$





The MPC controller considers all constraints on the actuator

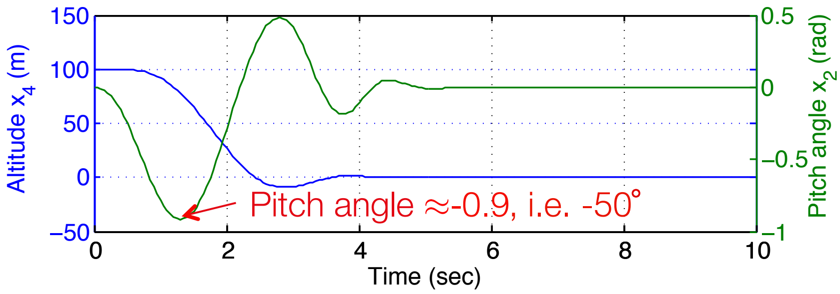
- Closed-loop system is stable
- Efficient use of the control authority

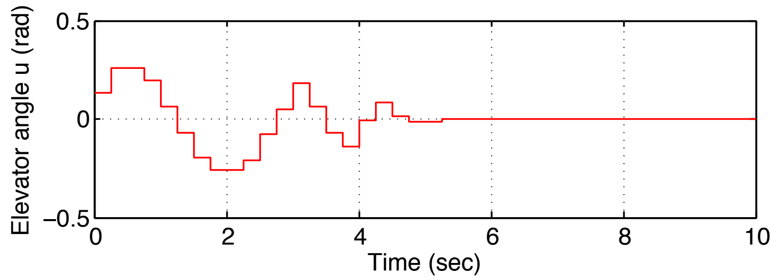
Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 10$





Increase step:
At time $t = 0$ the plane is flying with a deviation of 100m of the desired altitude, i.e. $x_0 = [0; 0; 0; 100]$

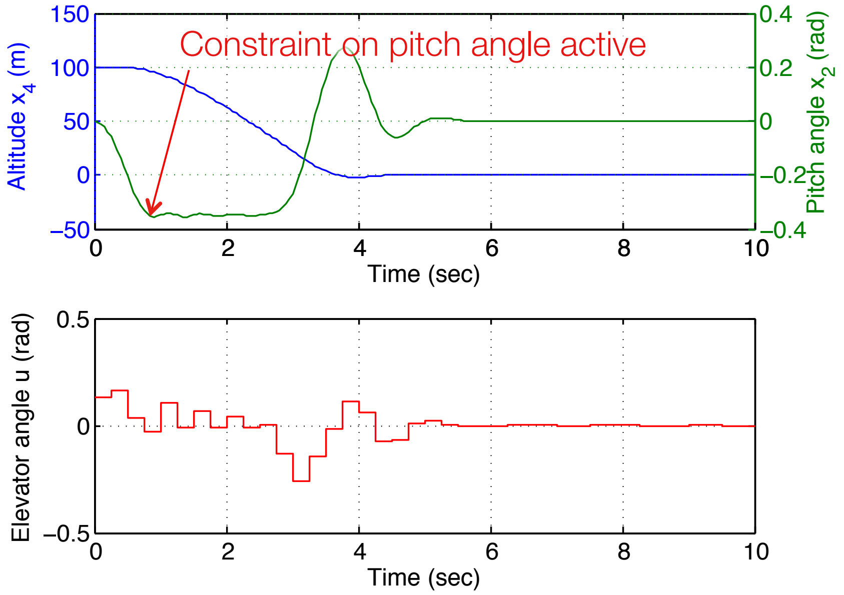
- Pitch angle too large during transient

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 10$



Add state constraints for passenger comfort:

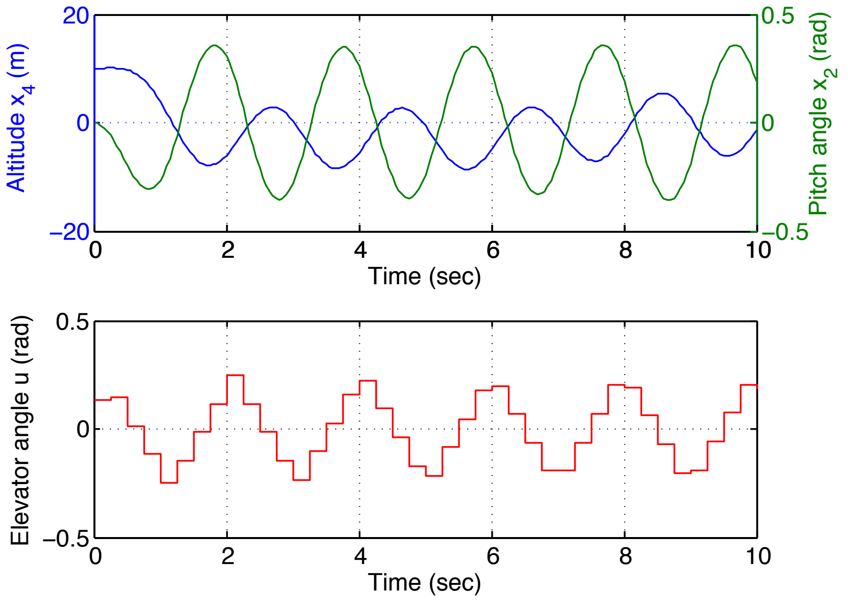
$$|x_2| \leq 0.349$$

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 4$



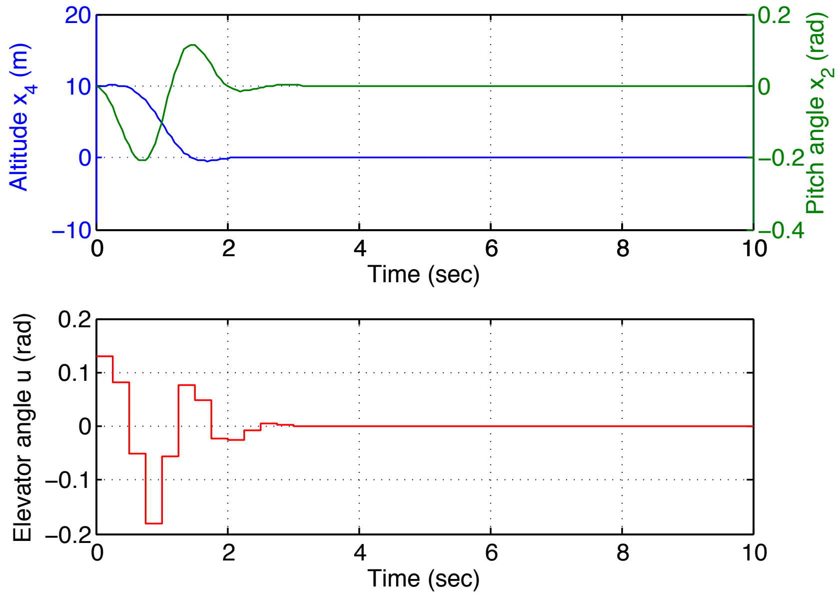
Decrease in the prediction horizon causes loss of the stability properties

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 4$



Inclusion of terminal cost and constraint provides stability

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Loss of Feasibility and Stability

What can go wrong with "standard" MPC?

- No feasibility guarantee, i.e., the MPC problem may not have a solution
- No stability guarantee, i.e., trajectories may not converge to the origin

Example: Loss of feasibility - Double Integrator

Consider the double integrator

$$\begin{cases} x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to the input constraints

$$-0.5 \leq u(t) \leq 0.5$$

and the state constraints

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Compute a receding horizon controller with quadratic objective with

$$N = 3,\ P = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = 10.$$

Example: Loss of feasibility - Double Integrator

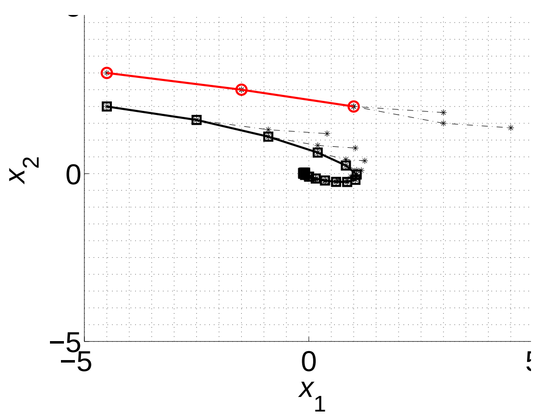
The QP problem associated with the RHC is

$$H = \begin{bmatrix} 13.50 & -10.00 & -0.50 \\ -10.00 & 22.00 & -10.00 \\ -0.50 & -10.00 & 31.50 \end{bmatrix}, \quad F = \begin{bmatrix} -10.50 & 10.00 & -0.50 \\ -20.50 & 10.00 & 9.50 \end{bmatrix}, \quad Y = \begin{bmatrix} 14.50 & 23.50 \\ 23.50 & 54.50 \end{bmatrix}$$

$$G_0 = \begin{bmatrix} 0.50 & -1.00 & 0.50 \\ -0.50 & 1.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ -0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & 0.50 \\ -1.00 & 0.00 & 0.00 \\ 0.00 & -1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 \\ -0.50 & 0.00 & 0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.50 & 0.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 0.50 & 0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ -0.50 & -0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 1.00 & 1.00 \\ -0.50 & -0.50 \\ -1.00 & -1.00 \\ 0.50 & 0.50 \\ -0.50 & -1.50 \\ 0.50 & 1.50 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.00 & -1.00 \end{bmatrix}, \quad w_0 = \begin{bmatrix} 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \end{bmatrix}$$

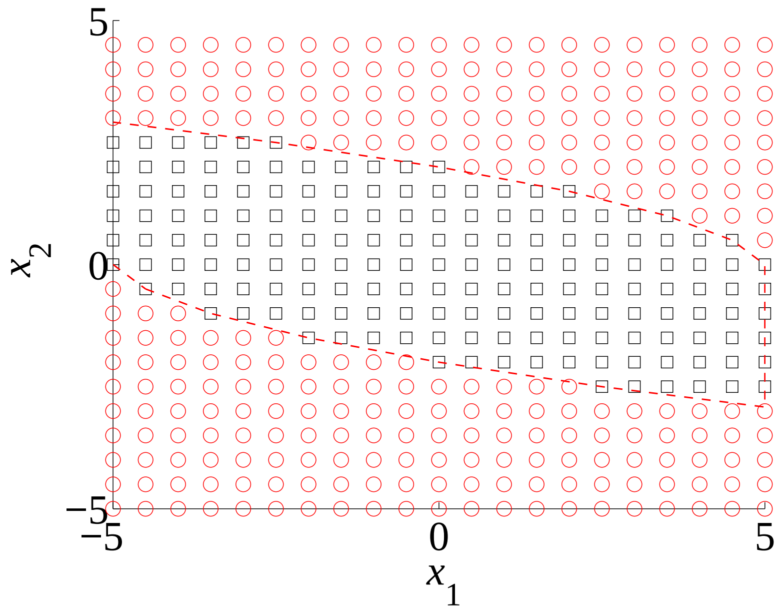
Example: Loss of feasibility - Double Integrator

1) MEASURE the state $x(t)$ at time instance t
2) OBTAIN $U_0^*(x(t))$ by solving the optimization problem in (1)
3) IF $U_0^*(x(t)) = \emptyset$ THEN 'problem infeasible' STOP
4) APPLY the first element u_0^* of U_0^* to the system
5) WAIT for the new sampling time $t+1$, GOTO 1)



Depending on initial condition, closed loop trajectory may lead to states for which optimization problem is infeasible.

Example: Loss of feasibility - Double Integrator



Boxes (*Circles*) are initial points leading (not leading) to feasible closed-loop trajectories

Example: Feasibility and stability are function of tuning

Unstable system $x(t+1) = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

Input constraints $-1 \leq u(t) \leq 1$

State constraints $\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

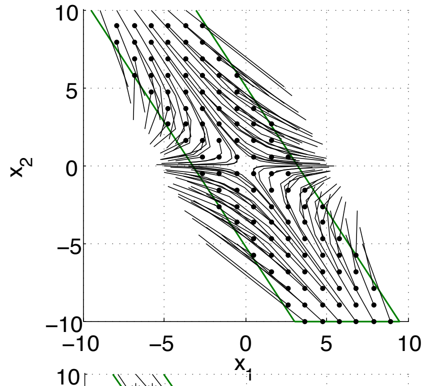
Parameters: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

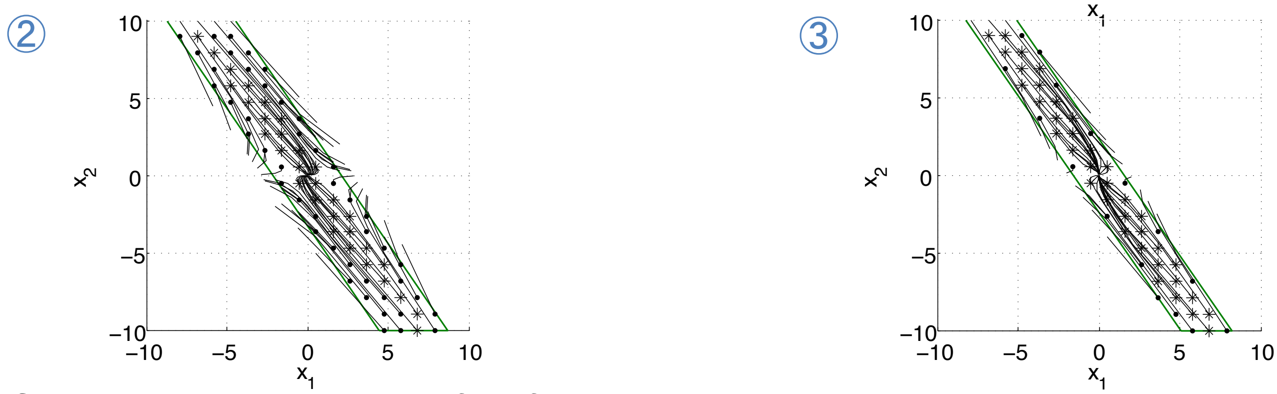
Investigate the stability properties for different horizons N and weights R by solving the finite-horizon MPC problem in a receding horizon fashion...

Example: Feasibility and stability are function of tuning

1. $R = 10,\ N = 2$: all trajectories unstable.
2. $R = 2,\ N = 3$: some trajectories stable.
3. $R = 1,\ N = 4$: more stable trajectories.

- $*$ Initial points with convergent trajectories
- $\circ$ Initial points that diverge



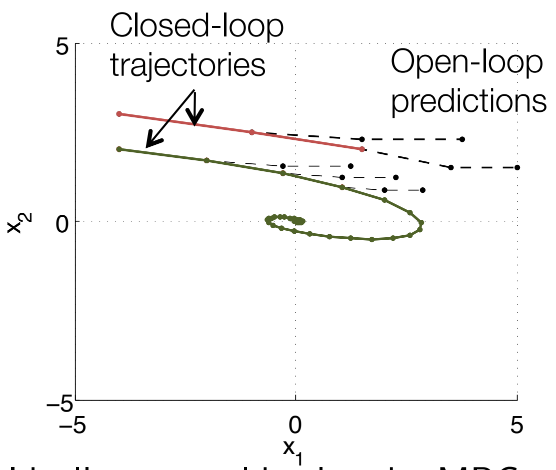


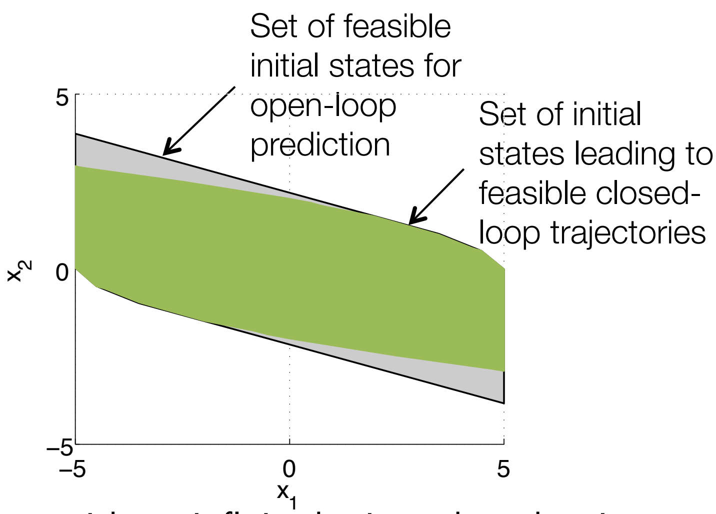
Green lines denote the set of all feasible initial points. They depend on the horizon N but not on the cost $R \Longrightarrow$ Parameters have complex effect and trajectories.

Summary: Feasibility and Stability

Problems originate from the use of a 'short sighted' strategy

$\Rightarrow$ Finite horizon causes deviation between the open-loop prediction and the closed-loop system:





Ideally we would solve the MPC problem with an infinite horizon, but that is computationally intractable

$\Rightarrow$ Design finite horizon problem such that it approximates the infinite horizon

Summary: Feasibility and Stability

- Infinite-Horizon
  If we solve the RHC problem for $N = \infty$ (as done for LQR), then the open loop trajectories are the same as the closed loop trajectories. Hence
  - If problem is feasible, the closed loop trajectories will be always feasible
  - If the cost is finite, then states and inputs will converge asymptotically to the origin

- Finite-Horizon
  RHC is "short-sighted" strategy approximating infinite horizon controller. But
  - **Feasibility**. After some steps the finite horizon optimal control problem may become infeasible. (Infeasibility occurs without disturbances and model mismatch!)
  - **Stability**. The generated control inputs may not lead to trajectories that converge to the origin.

Feasibility and stability in MPC - Solution

Main idea: Introduce terminal cost and constraints to explicitly ensure feasibility and stability:

$$
\begin{aligned}
J_0^*(x_0) = \quad & \min_{U_0} \quad p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) && \text{Terminal Cost} \\
& \text{subj. to} \\
& \quad x_{k+1} = Ax_k + Bu_k, \ k = 0, \ldots, N-1 \\
& \quad x_k \in \mathcal{X}, \ u_k \in \mathcal{U}, \ k = 0, \ldots, N-1 \\
& \quad x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& \quad x_0 = x(t)
\end{aligned}
$$

$p(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon.

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Feasibility and Stability of MPC: Proof

Main steps:

- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point
- Prove stability by showing that the optimal cost function is a Lyapunov function

Two cases:

1. Terminal constraint at zero: $x_N = 0$
2. Terminal constraint in some (convex) set: $x_N \in \mathcal{X}_f$

General notation:

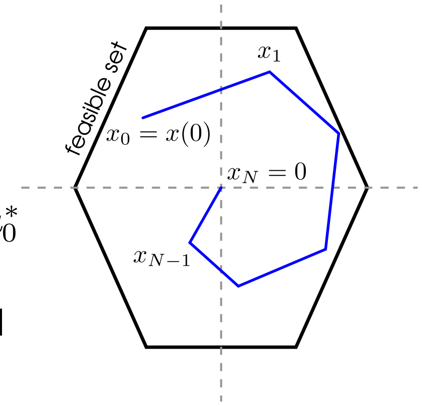
$$J_0^*(x_0) = \min_{U_0} \underbrace{p(x_N)}_{\text{terminal cost}} + \sum_{i=0}^{N-1} \underbrace{q(x_i, u_i)}_{\text{stage cost}}$$

Quadratic case: $q(x_i, u_i) = x_i^T Q x_i + u_i^T R u_i$, $p(x_N) = x_N^T P x_N$

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

- Assume feasibility of x_0 and let $\{u_0^*, u_1^*, \ldots, u_{N-1}^*\}$ be the optimal control sequence computed at x_0 and $\{x(0), x_1, \ldots, x_N\}$ be the corresponding state trajectory
- Apply u_0^* and let system evolve to $x(1) = Ax_0 + Bu_0^*$
- At $x(1)$ the control sequence $\{u_1^*, u_2^*, \ldots, u_{N-1}^*, 0\}$ is feasible (apply 0 control input $\Rightarrow x_{N+1} = 0$)



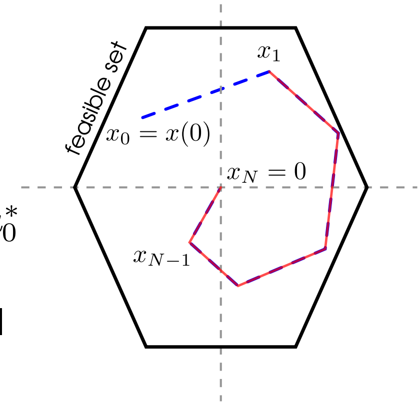
$\Rightarrow$ *Recursive feasibility* ✔

$\Rightarrow$ $J_0^*(x)$ *is a Lyapunov function* $\rightarrow$ *(Lyapunov) Stability* ✔

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

- Assume feasibility of x_0 and let $\{u_0^*,\ u_1^*,\ \ldots,\ u_{N-1}^*\}$ be the optimal control sequence computed at x_0 and $\{x(0),\ x_1,\ \ldots,\ x_N\}$ be the corresponding state trajectory
- Apply u_0^* and let system evolve to $x(1) = Ax_0 + Bu_0^*$
- At $x(1)$ the control sequence $\{u_1^*,\ u_2^*,\ \ldots,\ u_{N-1}^*,\ 0\}$ is feasible (apply 0 control input $\Rightarrow x_{N+1} = 0$)



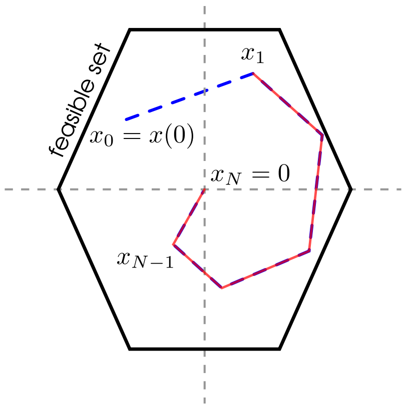
$\Rightarrow$ *Recursive feasibility* ✔

$\Rightarrow$ *$J_0^*(x)$ is a Lyapunov function $\to$ (Lyapunov) Stability* ✔

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

Goal: Show $J_0^*(x_1) < J_0^*(x_0) \quad \forall x_0 \neq 0$



$$J_0^*(x_0) = \underbrace{p(x_N)}_{=0} + \sum_{i=0}^{N-1} q(x_i, u_i^*)$$

$$\begin{aligned} J_0^*(x_1) \leq \tilde{J}_0(x_1) &= \sum_{i=1}^{N} q(x_i, u_i^*) \\ &= \sum_{i=0}^{N-1} q(x_i, u_i^*) - q(x_0, u_0^*) + q(x_N, u_N) \\ &= J_0^*(x_0) - \underbrace{q(x_0, u_0^*)}_{\substack{\text{Subtract cost} \\ \text{at stage } 0}} + \underbrace{q(0,0)}_{\substack{=0,\ \text{Add cost} \\ \text{for staying at } 0}} \end{aligned}$$

$\Rightarrow$ *$J_0^*(x)$ is a Lyapunov function $\to$ (Lyapunov) Stability* ✔

Example: Impact of Horizon with Zero Terminal Constraint

System dynamics:

$$x_{k+1} = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

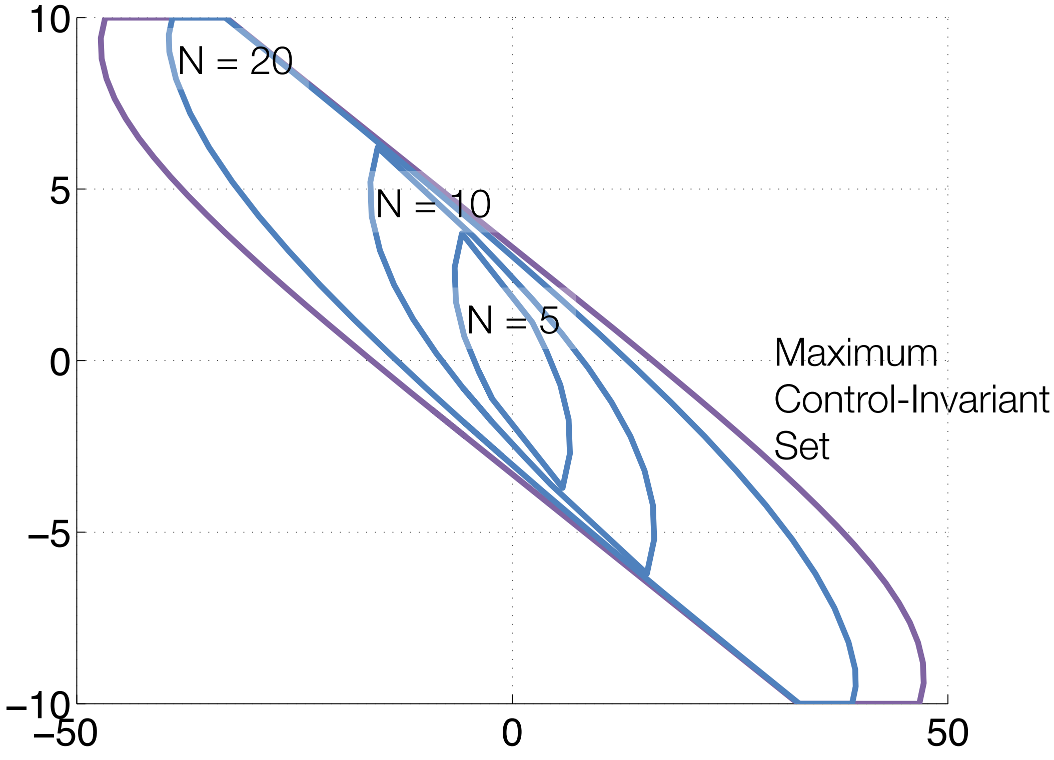
Constraints:

$$\mathcal{X} := \{x \mid -50 \le x_1 \le 50,\ -10 \le x_2 \le 10\} = \{x \mid A_x x \le b_x\}$$
$$\mathcal{U} := \{u \mid \|u\|_\infty \le 1\} = \{u \mid A_u u \le b_u\}$$

Stage cost:

$$q(x, u) := x' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^T u$$

Example: Impact of Horizon with Zero Terminal Constraint



The horizon can have a strong impact on the region of attraction.

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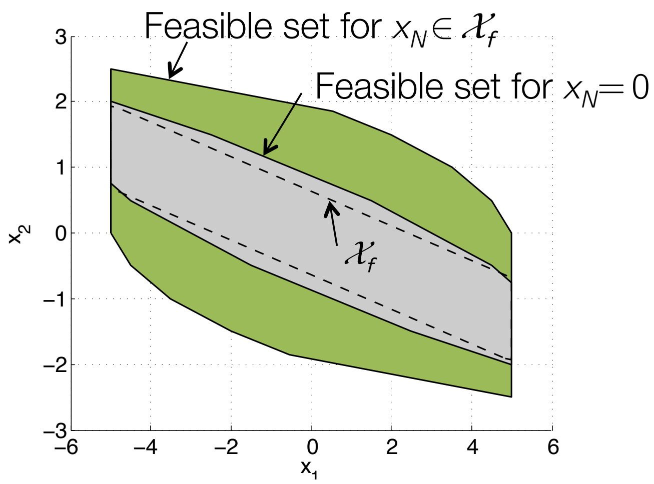
Extension to More General Terminal Sets

Problem: The terminal constraint $x_N = 0$ reduces the size of the feasible set

Goal: Use convex set $\mathcal{X}_f$ to increase the region of attraction



Double integrator

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$-0.5 \leq u(t) \leq 0.5$$

$$N = 5, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 10$$

Goal: Generalize proof to the constraint $x_N \in \mathcal{X}_f$

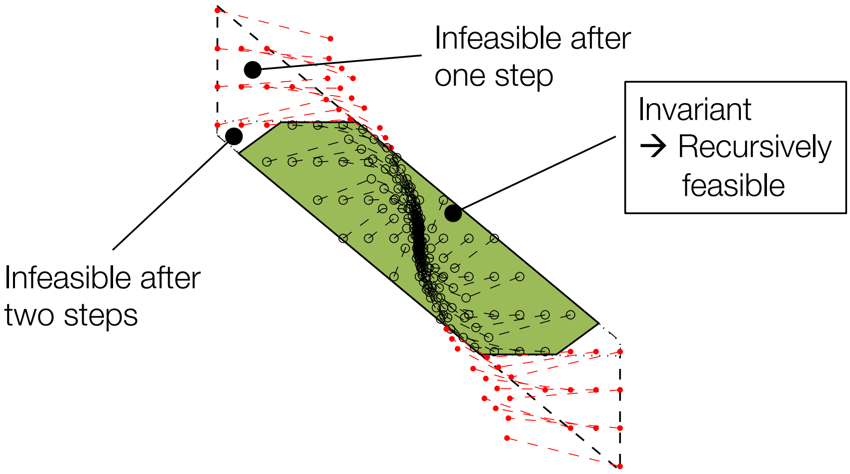
Invariant sets

Definition: Invariant set

A set $\mathcal{O}$ is called *positively invariant* for system $x(t+1) = f_{cl}(x(t))$, if

$$x(0) \in \mathcal{O} \Rightarrow x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}_+$$

The positively invariant set that contains every closed positively invariant set is called the maximal positively invariant set $\mathcal{O}_\infty$.



Stability of MPC - Main Result

Assumptions

1. Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2. Terminal set is **invariant** under the local control law $v(x_k)$:

$$x_{k+1} = Ax_k + Bv(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

   All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X}, \; v(x_k) \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$p(x_{k+1}) - p(x_k) \leq -q(x_k, v(x_k)), \quad \text{for all } x_k \in \mathcal{X}_f$$

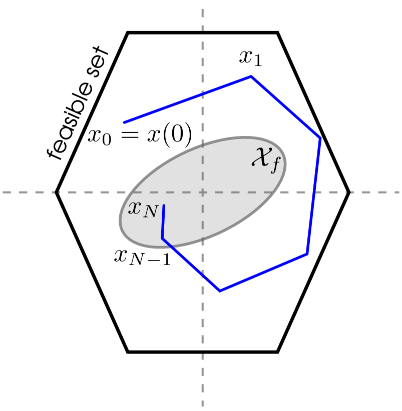
Under those 3 assumptions:

Theorem

The closed-loop system under the MPC control law $u_0^*(x)$ is asymptotically stable and the set $\mathcal{X}_f$ is positive invariant for the system $x(k+1) = Ax + Bu_0^*(x)$.

Stability of MPC - Outline of the Proof

- Assume feasibility of $x(0)$ and let $\{u_0^*,\ u_1^*,\ \ldots,\ u_{N-1}^*\}$ be the optimal control sequence computed at $x(0)$ and $\{x(0),\ x_1,\ \ldots,\ x_N\}$ the corresponding state trajectory



- At $x(1)$, $\{u_1^*,\ u_2^*,\ \ldots,\ v(x_N)\}$ is feasible:

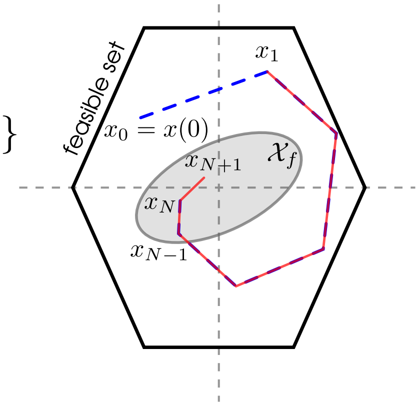
  x_N is in $\mathcal{X}_f \to v(x_N)$ is feasible

  and $x_{N+1} = Ax_N + Bv(x_N)$ in $\mathcal{X}_f$

$\Rightarrow$ *Terminal constraint provides recursive feasibility*

Stability of MPC - Outline of the Proof

- Assume feasibility of $x(0)$ and let $\{u_0^*,\ u_1^*,\ \ldots,\ u_{N-1}^*\}$ be the optimal control sequence computed at $x(0)$ and $\{x(0),\ x_1,\ \ldots,\ x_N\}$ the corresponding state trajectory



- At $x(1)$, $\{u_1^*,\ u_2^*,\ \ldots,\ v(x_N)\}$ is feasible:

x_N is in $\mathcal{X}_f \to v(x_N)$ is feasible

and $x_{N+1} = Ax_N + Bv(x_N)$ in $\mathcal{X}_f$

$\Rightarrow$ *Terminal constraint provides recursive feasibility*

Asymptotic Stability of MPC - Outline of the Proof

$$J_0^*(x_0) = \sum_{i=0}^{N-1} q(x_i, u_i^*) + p(x_N)$$

Feasible, sub-optimal sequence for x_1 : $\{u_1^*,\ u_2^*,\ \ldots,\ v(x_N)\}$

$$\begin{aligned} J_0^*(x_1) &\le \sum_{i=1}^{N} q(x_i, u_i^*) + p(Ax_N + Bv(x_N)) \\ &= \sum_{i=0}^{N-1} q(x_i, u_i^*) + p(x_N) - q(x_0, u_0^*) + p(Ax_N + Bv(x_N)) \\ &\qquad - p(x_N) + q(x_N, v(x_N)) \\ &= J_0^*(x_0) - q(x_0, u_0^*) + \underbrace{p(Ax_N + Bv(x_N)) - p(x_N) + q(x_N, v(x_N))}_{p(x) \le 0} \\ &\implies J_0^*(x_1) - J_0^*(x_0) \le -q(x_0, u_0^*), \quad q > 0 \end{aligned}$$

$J_0^*(x)$ is a Lyapunov function decreasing along the closed loop trajectories

$\Rightarrow$ The closed-loop system under the MPC control law is asymptotically stable

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

$$
\begin{aligned}
J_0^*(x_0) = \quad & \min_{U_0} \quad x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k && \text{Terminal Cost} \\
& \text{subj. to} \\
& \quad x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\
& \quad x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& \quad x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& \quad x_0 = x(t)
\end{aligned}
$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

- Design unconstrained LQR control law

$$F_\infty = -(B' P_\infty B + R)^{-1} B' P_\infty$$

where P_∞ is the solution to the discrete-time algebraic Riccati equation:

$$P_\infty = A' P_\infty A + Q - A' P_\infty B (B' P_\infty B + R)^{-1} B' P_\infty A$$

- Choose the terminal weight $P = P_\infty$
- Choose the terminal set $\mathcal{X}_f$ to be the maximum invariant set for the closed-loop system $x_{k+1} = (A + BF_\infty)x_k$:

$$x_{k+1} = Ax_k + BF\infty(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X},\ F_\infty x_k \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

1. The stage cost is a positive definite function
2. By construction the terminal set is **invariant** under the local control law $v = F_\infty x$
3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$\begin{aligned} x'_{k+1}Px_{k+1} - x'_kPx_k =& x'_k(-P_\infty + A'P_\infty A - A'P_\infty B(B'P_\infty B + R)^{-1}B'P_\infty A)x_k \\ =& -x'_kQx_k \end{aligned}$$

All the Assumptions of the Feasibility and Stability Theorem are verified.

Example: Unstable Linear System

System dynamics:

$$x_{k+1} = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

Constraints:

$$\begin{aligned} &\mathcal{X} := \{x \mid -50 \le x_1 \le 50, \ -10 \le x_2 \le 10\} = \{x \mid A_x x \le b_x\} \\ &\mathcal{U} := \{u \mid \|u\|_\infty \le 1\} = \{u \mid A_u u \le b_u\} \end{aligned}$$

Stage cost:

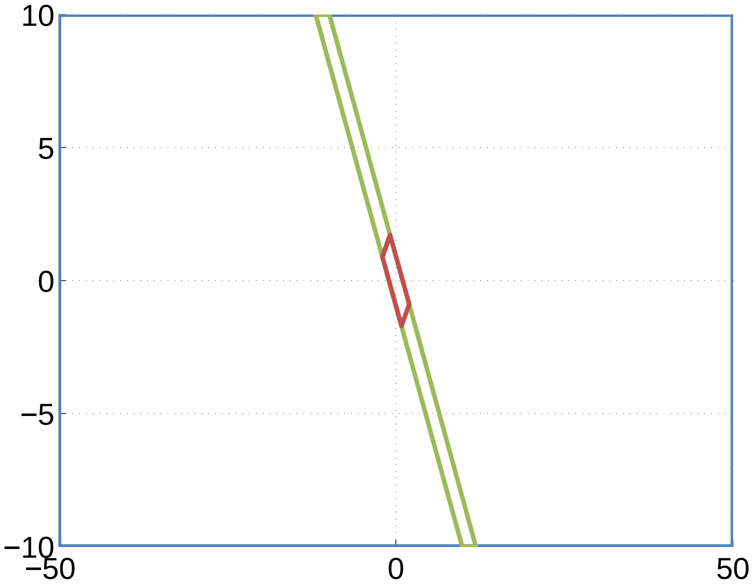
$$q(x,u) := x' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^T u$$

Horizon: $N = 10$

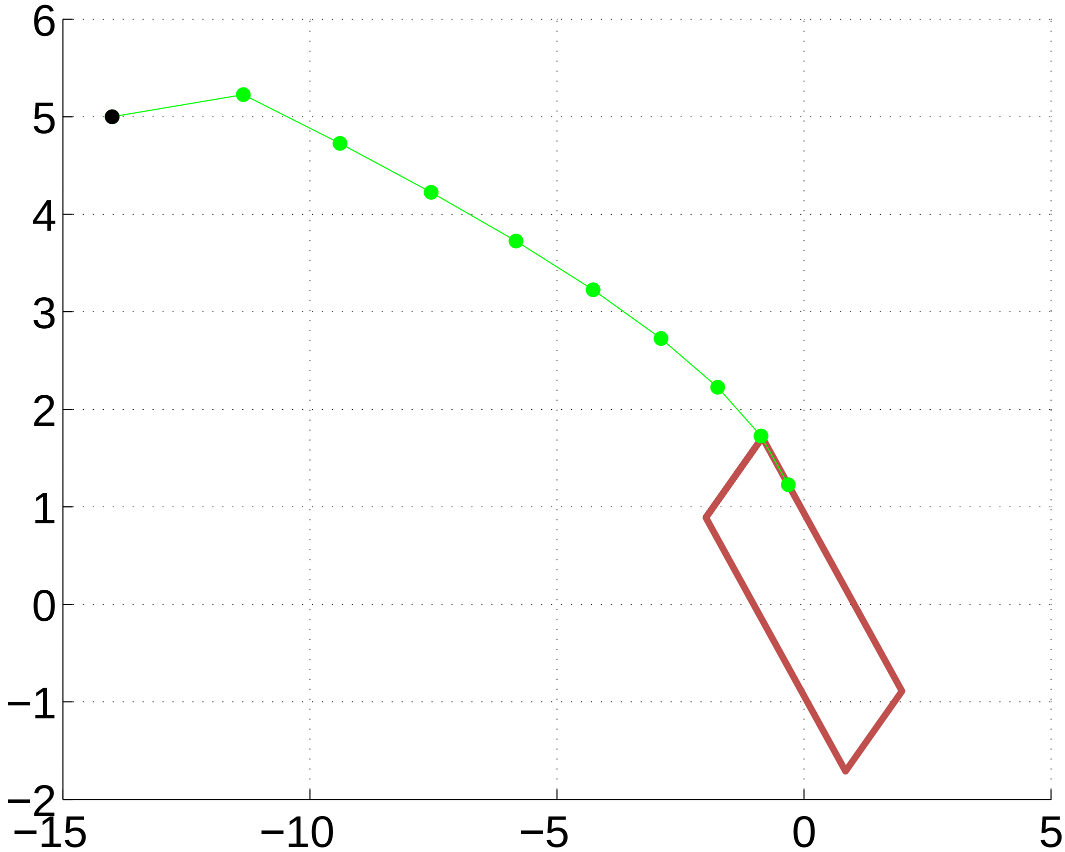
Example: Designing MPC Problem

1. Compute the optimal LQR controller and cost matrices: F_∞, P_∞
2. Compute the maximal invariant set $\mathcal{X}_f$ for the closed-loop linear system $x_{k+1} = (A + BF_\infty)x_k$ subject to the constraints

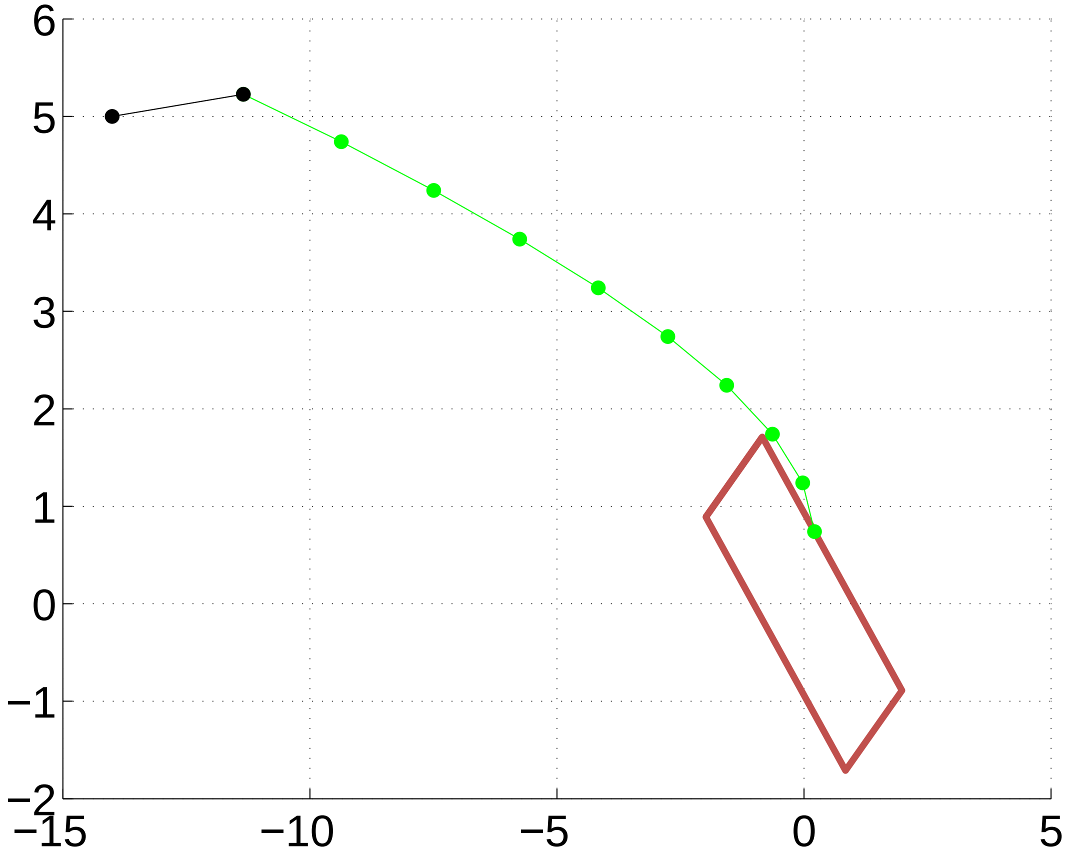
$$\mathcal{X}_{\text{cl}} := \left\{ x \;\middle|\; \begin{bmatrix} A_x \\ A_u F_\infty \end{bmatrix} x \le \begin{bmatrix} b_x \\ b_u \end{bmatrix} \right\}$$



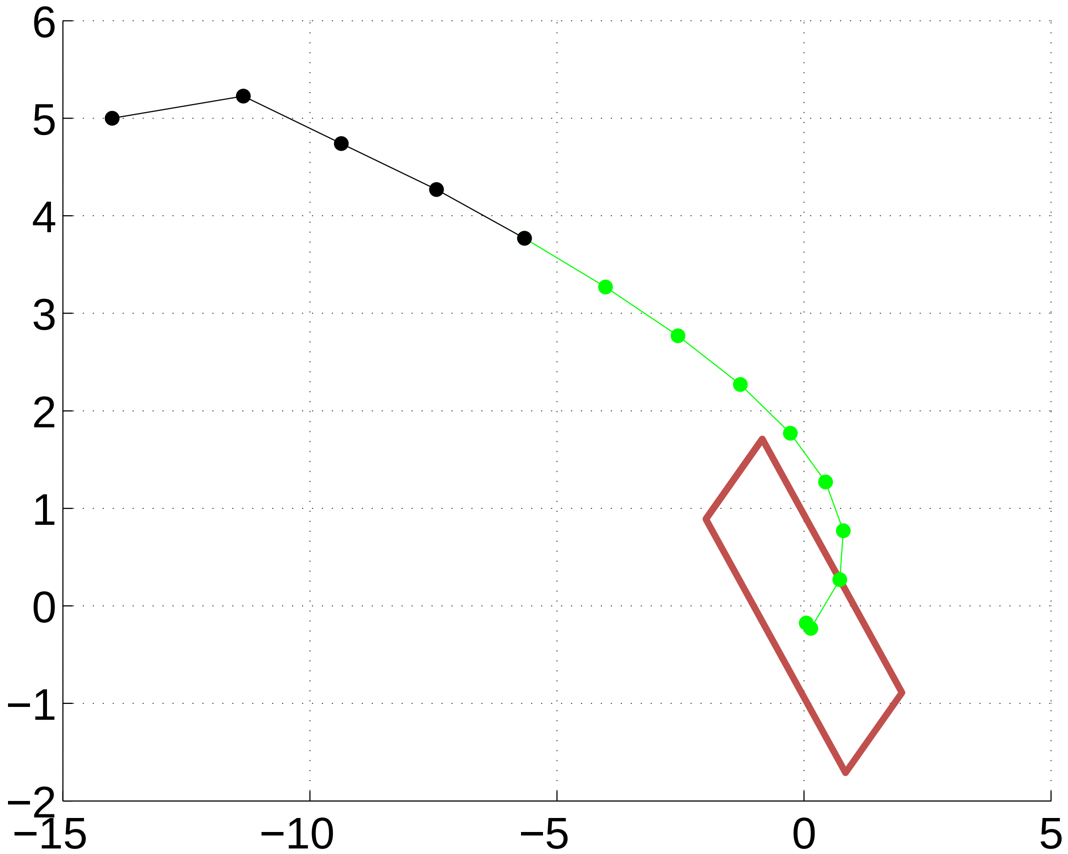
Example: Closed-loop behaviour



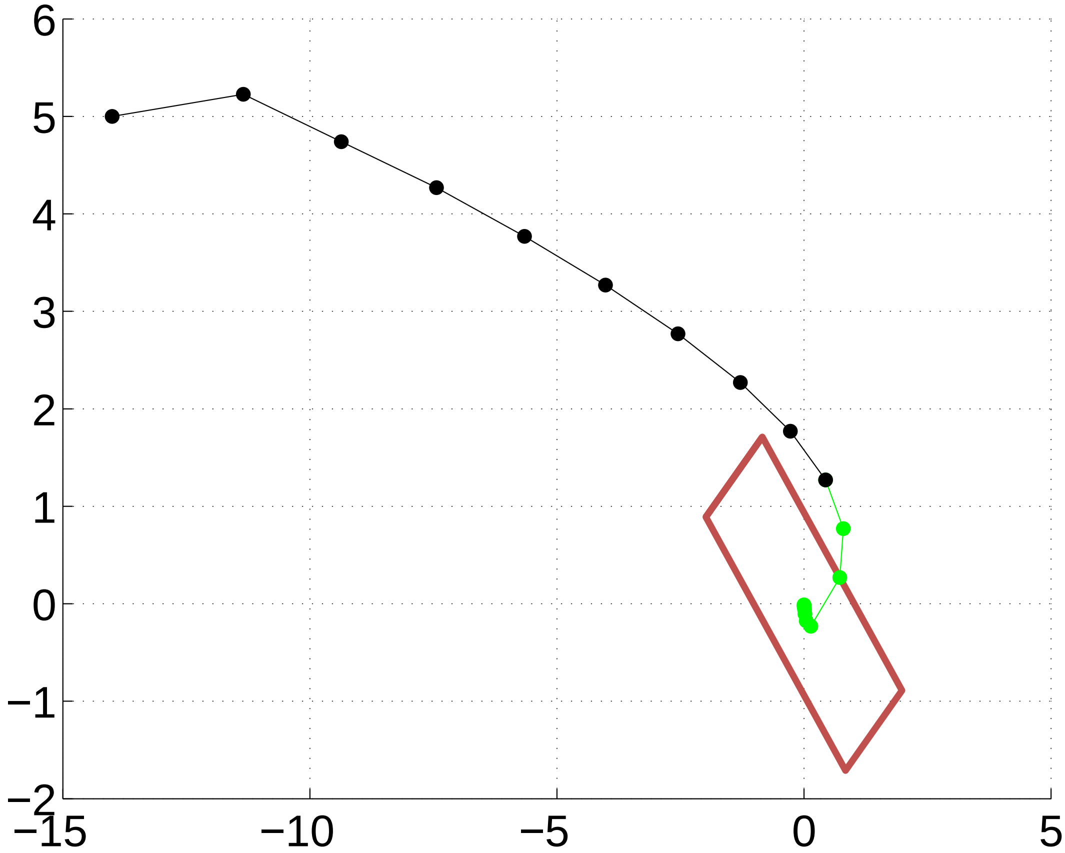
Example: Closed-loop behaviour



Example: Closed-loop behaviour

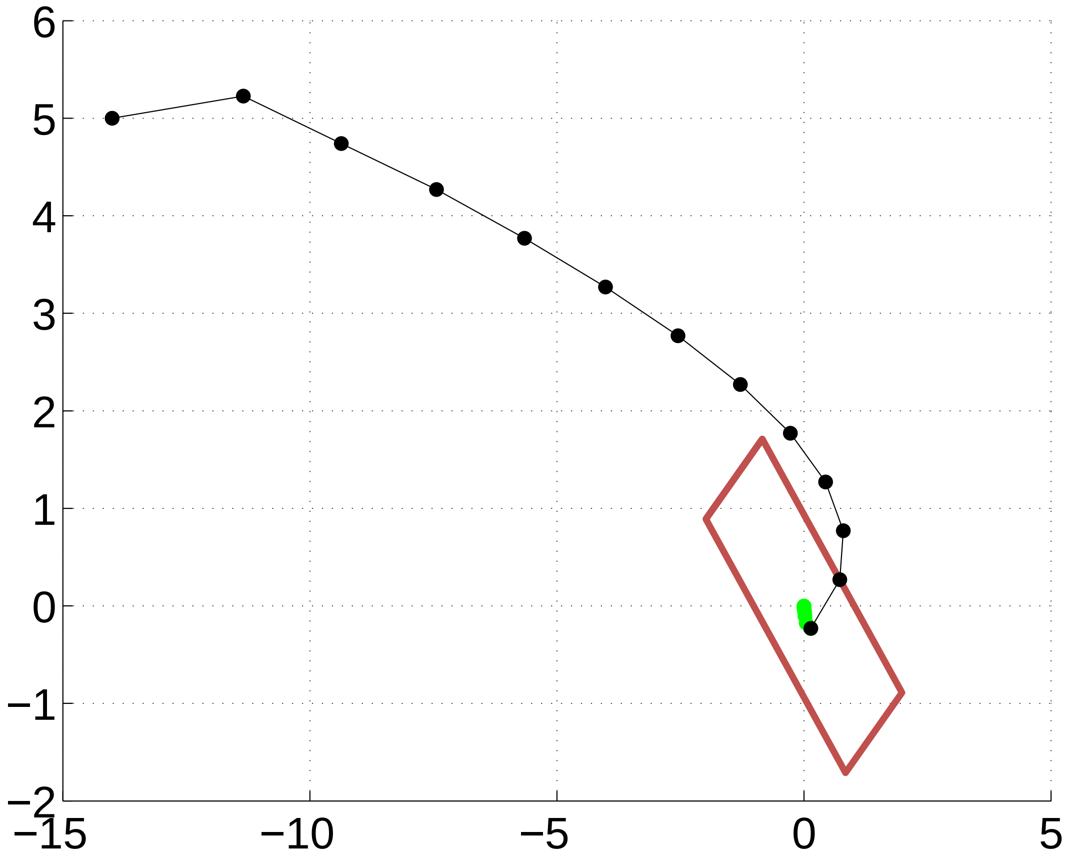


Example: Closed-loop behaviour



6 6. Feasibility and Stability 6.2 General Terminal Sets

Example: Closed-loop behaviour



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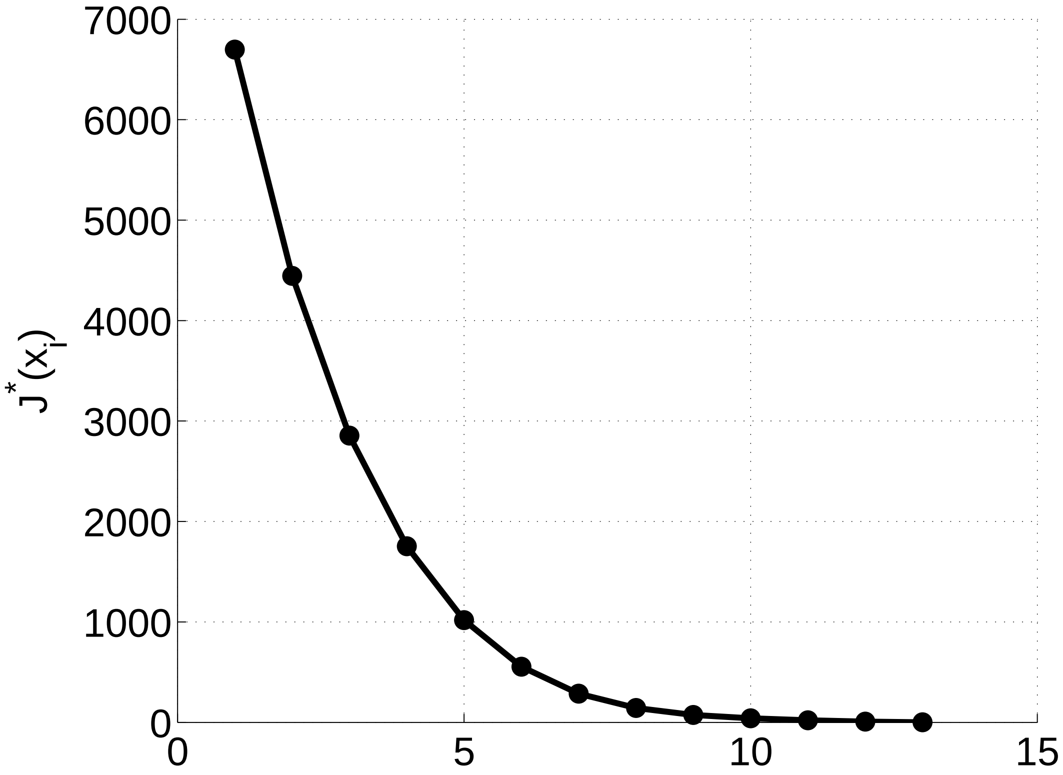
Example: Lyapunov Decrease of Optimal Cost



Stability of MPC - Remarks

- The terminal set $\mathcal{X}_f$ and the terminal cost ensure recursive feasibility and stability of the closed-loop system.
  But: the terminal constraint reduces the region of attraction.
  (Can extend the horizon to a sufficiently large value to increase the region)

Are terminal sets used in practice?

- Generally not...
  - Not well understood by practitioners
  - Requires advanced tools to compute (polyhedral computation or LMI)
- Reduces region of attraction
  - A 'real' controller must provide *some* input in *every* circumstance
- Often unnecessary
  - Stable system, long horizon $\rightarrow$ will be stable and feasible in a (large) neighbourhood of the origin

Choice of Terminal Set and Cost: Summary

- Terminal constraint provides a sufficient condition for stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attaction for small N
- Solution for linear systems with quadratic cost
- In practice: Enlarge horizon and check stability by sampling
- With larger horizon length N, region of attraction approaches maximum control invariant set

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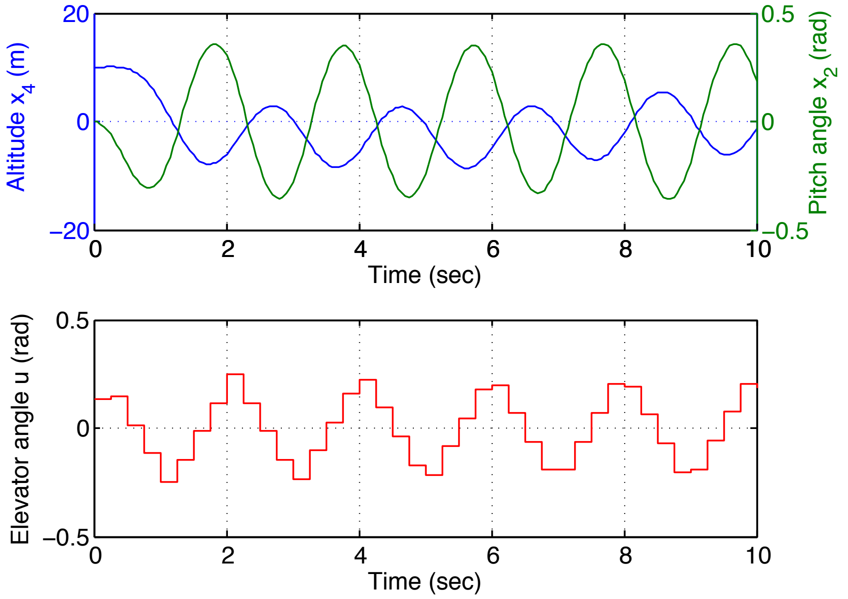
Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$
and rate constraints $|\dot{u}_i| \leq 0.349$
approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec,
$Q = I,\ R = 10,\ N = 4$



Decrease in the prediction horizon causes loss of the stability properties

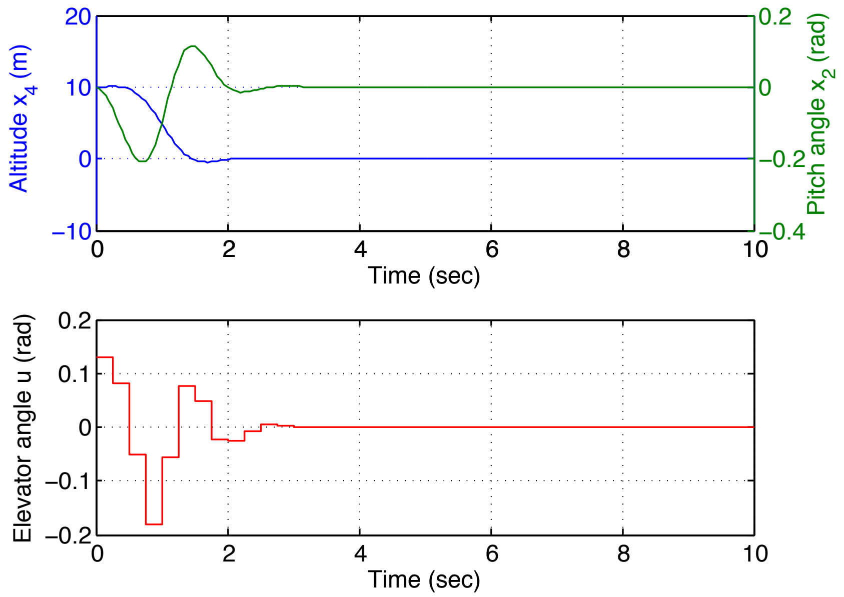
6 6. Feasibility and Stability 6.3 Example

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$
and rate constraints $|\dot{u}_i| \leq 0.349$
approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec,
$Q = I,\ R = 10,\ N = 4$



Inclusion of terminal cost and constraint provides stability

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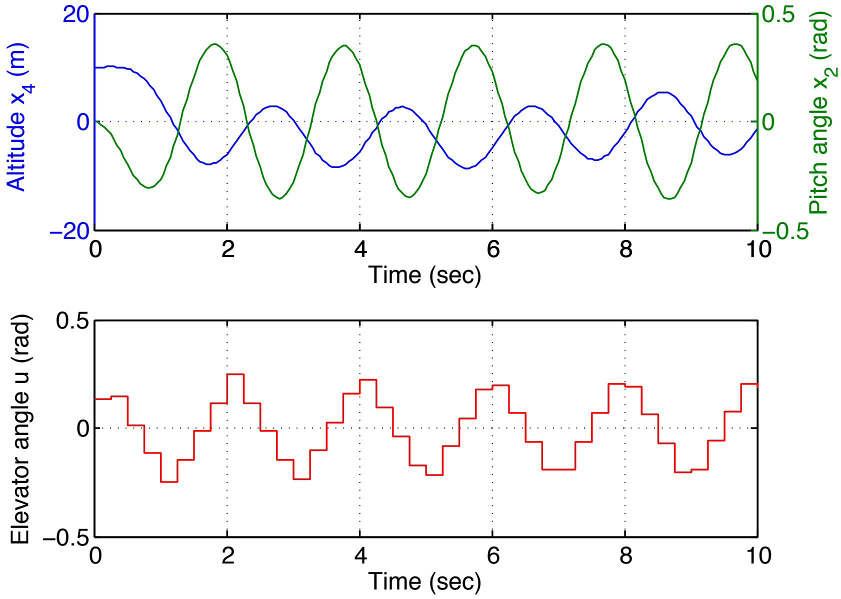
Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 4$



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Example: Short horizon

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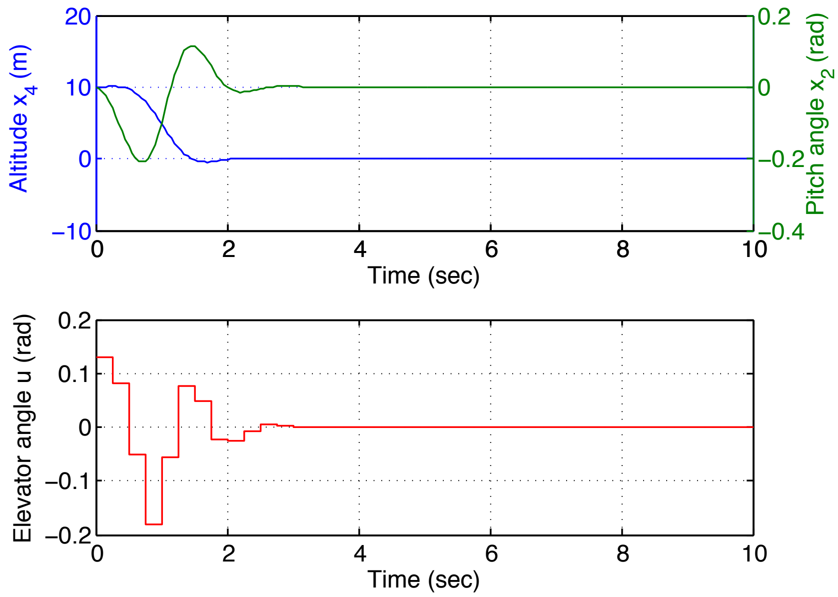

Inclusion of terminal cost and constraint provides stability

Summary

Finite-horizon MPC may not be stable!

Finite-horizon MPC may not satisfy constraints for all time!

- An infinite-horizon provides stability and invariance.
- We ‘fake’ infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- These ideas extend to non-linear systems, but the sets are difficult to compute.

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Extension to Nonlinear MPC

Consider the nonlinear system dynamics: $x(t+1) = g(x(t), u(t))$

$$\begin{aligned} J_0^*(x(t)) = \quad & \min_{U_0} && p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\ & \text{subj. to} && x_{k+1} = g(x_k, u_k),\ k = 0, \ldots, N-1 \\ & && x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & && x_N \in \mathcal{X}_f \\ & && x_0 = x(t) \end{aligned}$$

- Presented assumptions on the terminal set and cost did not rely on linearity
- Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems

→ Results can be directly extended to nonlinear systems.

However, computing the sets $\mathcal{X}_f$ and function p can be very difficult!