Chapter 4

State feedback and Observer Feedback

4.1 Pole placement via state feedback

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x \in \Re^n,\ u \in \Re \\ y &= Cx + Du \end{aligned}$$

- Poles of transfer function are eigenvalues of A
- Pole locations affect system response
  - stability
  - convergence rate
  - command following
  - disturbance rejection
  - noise immunity ....
- Assume $x(t)$ is available
- Design $u = -Kx + v$ to affect closed loop eigenvalue:

  $$\dot{x} = Ax + B(-Kx + v) = \underbrace{(A - BK)}_{Ac} x + Bv$$

  such that eigenvalues of A_c are $\sigma_1, \ldots, \sigma_n$.
- K = state feedback gain; v = auxiliary input.

4.2 Controller Canonical Form (SISO)

A system is said to be in **controller (canonical) form** if:

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}}_{A} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

What is the relationship between a_i, $i = 0, \ldots, n-1$ and eigenvalues of A?

- Consider the characteristic equation of A:

$$\psi(s) = det(sI - A) = det \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_0 & a_1 & a_2 \end{pmatrix}$$

- Eigenvalues of A, $\lambda_1, \ldots, \lambda_n$ are roots of $\psi(s) = 0$.

$$\psi(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

- Therefore, **if we can arbitrarily choose $a_0, \ldots, a_{n-1}$, we can choose the eigenvalues of A.**

Target characteristic polynomial

- Let desired eigenvalues be σ_1, σ_2, $\ldots \sigma_n$.
- Desired characteristic polynomial:

$$\bar{\psi}(s) = \Pi_{i=1}^{n}(s - \sigma_i) = s^n + \bar{a}_{n-1}s^{n-1} + \ldots + \bar{a}_1 s + \bar{a}_0$$

Some properties of characteristic polynomial for its proper design:

- If σ_i are in conjugate pair (i.e. for complex poles, $\alpha \pm j\beta$), then $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{n-1}$ are real numbers; and vice versa.
- Sum of eigenvalues: $\bar{a}_{n-1} = -\sum_{i=1}^{n} \sigma_i$
- Product of eigenvalues: $\bar{a}_0 = (-1)^n \Pi_{i=1}^{n} \sigma_i$
- If $\sigma_1, \ldots, \sigma_n$ all have negative real parts, then $\bar{a}_i > 0$ $i = 0, \ldots n-1$.
- If any of the polynomial coefficients is non-positive (negative or zero), then one or more of the roots have nonnegative real parts.

Consider state feedback:

$$u = -Kx + v$$
$$K = [k_0,\ k_1,\ k_2]$$

Closed loop equation:

$$\dot{x} = Ax + B(-Kx + v) = \underbrace{(A - BK)}_{Ac} x + Bv$$

with

$$A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0 + k_0) & -(a_1 + k_1) & -(a_2 + k_2) \end{pmatrix}$$

Thus, to place poles at $\sigma_1, \ldots, \sigma_n$, choose

$$\bar{a}_0 = a_0 + k_0 \Rightarrow k_0 = \bar{a}_0 - a_0$$
$$\bar{a}_1 = a_1 + k_1 \Rightarrow k_1 = \bar{a}_1 - a_1$$
$$\vdots$$
$$\bar{a}_{n-1} = a_{n-1} + k_{n-1} \Rightarrow k_{n-1} = \bar{a}_{n-1} - a_{n-1}$$

4.3 Conversion to controller canonical form

$$\dot{x} = Ax + Bu$$

- If we can convert a system into controller canonical form via invertible transformation $T \in \Re^{n\times n}$:

$$z = T^{-1}x; \quad A_z = T^{-1}AT, \quad B_z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = T^{-1}B$$

  where $\dot{z} = A_z z + B_z u$ is in controller canonical form:

$$A_z = \begin{pmatrix} 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ 0 & \ldots & 0 & 1 \\ -a_0 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix} \qquad B_z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

  we can design state feedback

$$u = -K_z z + v$$

  to place the poles (for the transformed system).

- Since A and $A_z = T^{-1}AT$ have same characteristic polynomials:

$$\begin{aligned} det(\lambda I - T^{-1}AT) &= det(\lambda T^{-1}T - T^{-1}AT) \\ &= det(T)det(T^{-1})det(\lambda I - A) \\ &= det(\lambda I - A) \end{aligned}$$

The control law:

$$u = -K_z T^{-1} x + v = -Kx + v$$

where $K = K_z T^{-1}$ places the poles at the desired locations.

Theorem *For the single input LTI system, $\dot{x} = Ax + Bu$, there is an invertible transformation T that converts the system into controller canonical form if and only if the system is controllable.*
Proof:

- "Only if": If the system is not controllable, then using Kalman decomposition, there are modes that are not affected by control. Thus, eigenvalues associated with those modes cannot be changed. This means that we cannot transform the system into controller canonical form, since otherwise, we can arbitrarily place the eigenvalues.

- "If": Let us construct T. Take $n = 3$ as example, and let T be:

$$T = [v_1 \mid v_2 \mid v_3]$$

$$A = T \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} T^{-1}; \qquad B = T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This says that $v_3 = B$.

Note that A_z is determined completely by the characteristic equation of A.

$$AT = T \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} \tag{4.1}$$

Now consider each column of (4.1) at a time, starting from the last. This says that:

$$A \cdot v_3 = v_2 - a_2 v_3, \Rightarrow \;\; v_2 = Av_3 + a_2 v_3 = AB + a_2 B$$

Having found v_2, we can find v_1 from the 2nd column from (4.1). This says,

$$\begin{aligned} & A \cdot v_2 = v_1 - a_1 v_3, \\ \Rightarrow \quad & v_1 = Av_2 + a_1 v_3 = A^2 B + a_2 AB + a_1 B \end{aligned}$$

- Now we check if the first column in (4.1) is consistent with the v_1, v_2 and v_3 we had found. It says:

$$A \cdot v_1 + a_0 v_3 = 0.$$

Is this true? The LHS is:

$$\begin{aligned} A \cdot v_1 + a_o v_3 =& A^3 B + a_2 A^2 B + a_1 AB + a_0 B \\ =& (A^3 + a_2 A^2 + a_1 A + a_0 I) B \end{aligned}$$

Since $\psi(s) = s^3 + a_2 s^2 + a_1 s + a_0$ is the characteristic polynomial of A, by the Cayley Hamilton Theorem, $\psi(A) = 0$, so $A^3 + a_2 A^2 + a_1 A + a_0 I = 0$. Hence, $A \cdot v_1 + a_0 v_3 = 0$.

- To complete the proof, we need to show that if the system is controllable, then T is non-singular. Notice that

$$T = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} B & AB & A^2B \end{pmatrix} \begin{pmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that T is non-singular if and only if the controllability matrix is non-singular. $\diamond$

Summary procedure for pole placement:

- Find characteristic equation of A,

$$\psi_A(s) = det(sI - A)$$

- Define the target closed loop characteristic equation $\psi_{A_c}(s) = \Pi_{i=1}^{n}(s - \sigma_i)$, where σ_i are the desired pole locations.

- Compute v_n, v_{n-1} etc. successively to contruct T,

$$\begin{aligned} v_n &= b \\ v_{n-1} &= Av_n + a_{n-1}b \\ &\vdots \\ v_k &= Av_{k+1} + a_k b \end{aligned}$$

- Find state feedback for transformed system: $z = T^{-1}x$:

$$u = K_z z + v$$

- Transform the feedback gain back into original coordinates:

$$u = Kx + v; \qquad K = K_z T^{-1}.$$

Example:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 2 & -2 \\ 1 & -2 & 4 \\ -5 & -1 & 3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u \qquad x(0) = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x \end{aligned}$$

The open loop system is unstable, as A has eigenvalues -4.4641, 2.4641 and 2. The system response to a unit step input is shown in Fig. 4.1.

- Characteristic equation of A

$$\psi_A(s) = s^3 - 15s + 22 = 0$$

- Desired closed loop pole locations : $(-2, -3, -4)$. Closed loop characteristic equation

$$\psi_{A_c}(s) = (s + 2)(s + 3)(s + 4) = s^3 + 9s^2 + 26s + 24$$

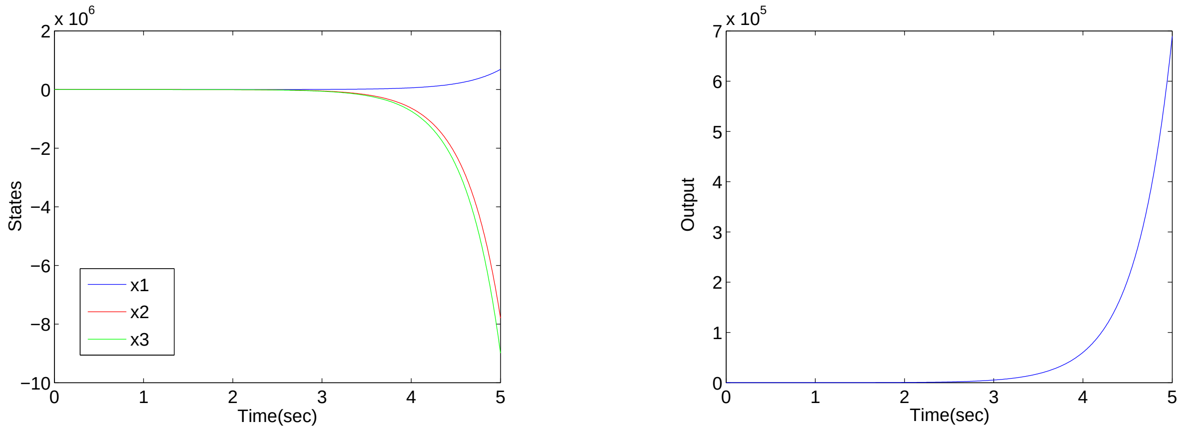

Figure 4.1: Open loop response of system to step input

- $T = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. Using the procedure detailed above,

$$v_3 = B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} \qquad v_1 = \begin{pmatrix} -2 \\ -23 \\ -11 \end{pmatrix}$$

$$T = \begin{pmatrix} -2 & -1 & 1 \\ -23 & 1 & 0 \\ -11 & -5 & 0 \end{pmatrix}$$

- Comparing the coefficients of ψ_A and ψ_{A_c}, we obtain the gain vector K_z in transformed coordinates

$$K_z = \begin{pmatrix} 2 & 41 & 9 \end{pmatrix}$$

- Transform K_z into original system.

$$K = K_z T^{-1} \quad = \begin{pmatrix} 9.0000 & 3.5714 & -9.2857 \end{pmatrix}$$

- Feedback control

$$u = -Kx + v$$

where v is an exogenous input. In our example, $v = 1$ is the reference input, and the feedback law is implemented as:

$$u = -Kx + gv$$

where g is a gain to be selected for reference tracking. Here it is selected as -11.98 to compensate for the DC gain. As seen from Fig. 4.2, the output y reaches the target value of 1.

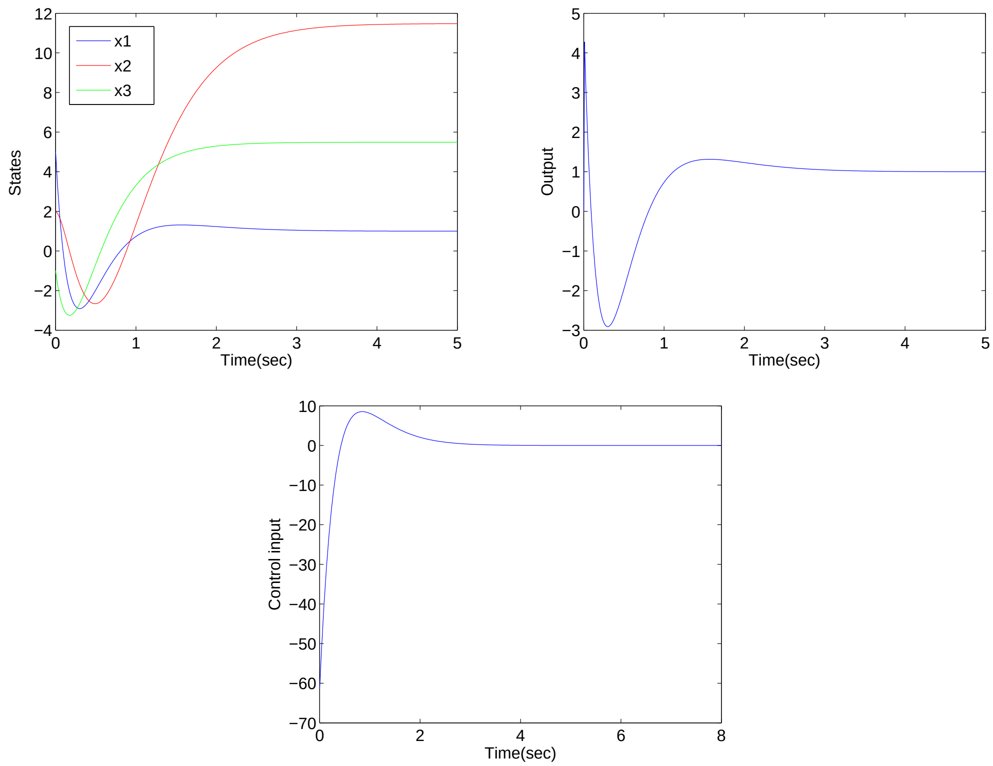

Figure 4.2: Closed loop response of system to step input

Arbitrarily pole placement???

Consider system

$$\dot{x} = -x + u$$
$$y = x$$

Let's place pole at $s = -100$ and match the D.C. gain.

Consider $u = -Kx + 100v$. The using $K = 99$,

$$\dot{x} = -(1 + K)x + 100v = -100(x - v).$$

This gives a transfer function of

$$X(s) = \frac{100}{s + 100} V(s).$$

If $v(t)$ is a step input, and $x(0) = 0$, then $u(0) = 100$ which is very large. Most likely saturates the system.

Thus, due to physical limitations, it is not practically possible to achieve arbitrarily fast eigen values.

4.4 Pole placement - multi-input case

$$\dot{x} = Ax + Bu$$

with $B \in \Re^{n \times m}$, $m > 1$.

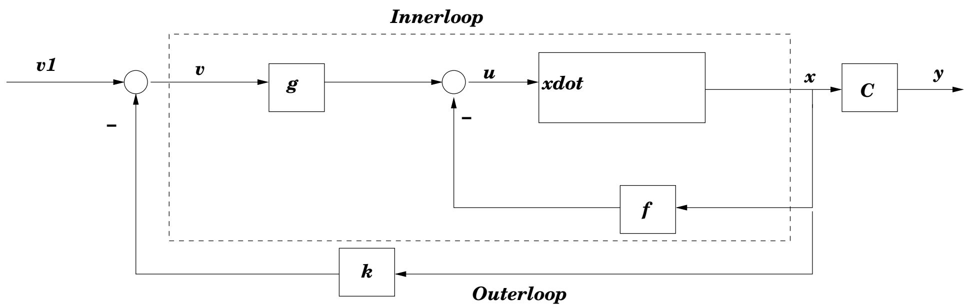

Figure 4.3: Hautus-Keymann Lemma

- The choice of eigenvalues do not uniquely specify the feedback gain K.
- Many choices of K lead to same eigenvalues but different eigenvectors.
- Possible to assign eigenvectors in addition to eigenvalues.

Hautus Keymann Lemma

Let (A, B) be controllable. Given any $b \in Range(B)$, there exists $F \in \Re^{m\times n}$ such that $(A - BF, b)$ is controllable.

Suppose that $b = B \cdot g$, where $g \in \Re^m$.

- Inner loop control:

$$u = -Fx + gv \quad \Rightarrow \dot{x} = (A - BF)x + bv$$

- Outer loop SI control:

$$v = -kx + v_1$$

  where k is designed for pole-placement (using technique previously given). See Fig. 4.3.

  We can write

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= Ax + B(-Fx + gv) \\ &= (A - BF)x + bv \\ &= A_1 x + b(-kx + v_1) \\ &= (A_1 - bk)x + bv_1 \\ &= A_{CL}x + bv_1\end{aligned}$$

  Design k so that closed loop poles, ie, eigen values of A_{CL} are placed at desired locations.

- It is interesting to note that generally, it may not be possible to find a $b \in \Re^n \in Range(B)$ such that (A, b) is controllable. For example: for

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  we cannot find a $g \in \Re^m$ such that (A, Bg) is controllable. We can see this by applying the PBH test with $\lambda = 2$. The reason for this is that A has repeated eigenvalues at $\lambda = 2$ with more than one independent eigenvector. The same situation applies, if A is semi-simple, with repeated eigenvalues.

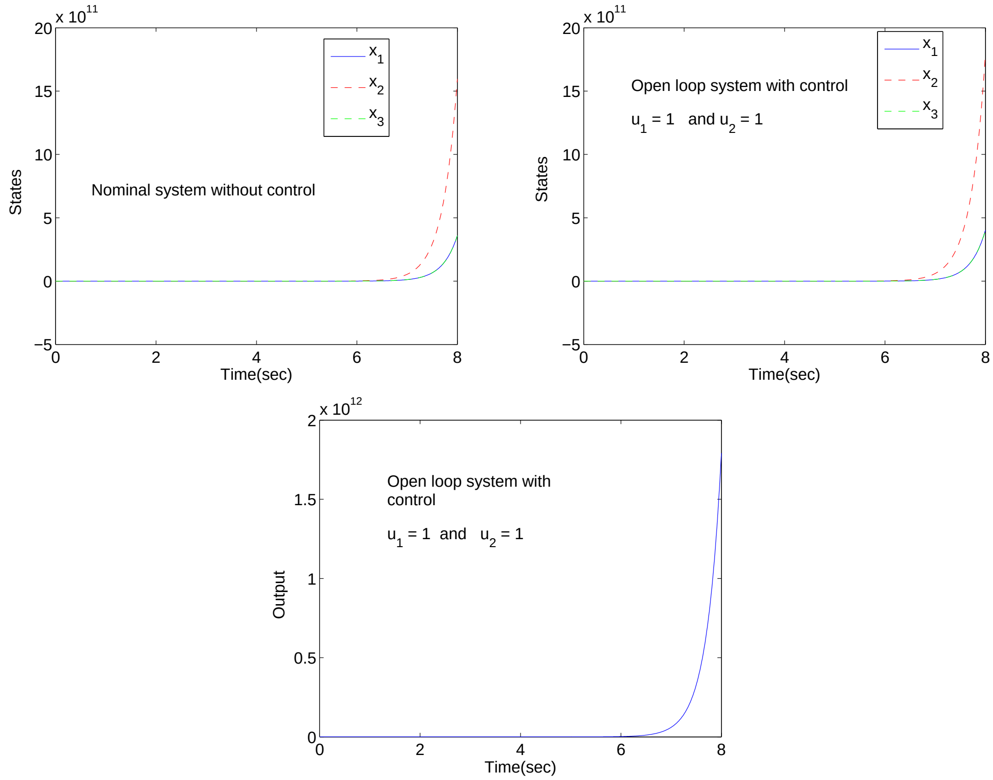

Figure 4.4: Nominal system, and open loop control

- What the Hautus Keymann Theorem says is that it is possible after preliminary state feedback using a matrix F. In fact, generally, most F matrices will make $\bar{A} = A - BF$ has distinct eigenvalues. This makes it possible to avoid the parasitic situation mentioned above, so that one can find $g \in \Re^m$ so that $(\bar{A}, Bg)$ is controllable.

Generally, eigenvalue assignment for a multiple input system is not unique. There will be some possibilities of choosing the eigenvectors also. However, for the purpose of this class, we shall use the optimal control technique to resolve the issue of choosing appropriate feedback gain K in $u = -Kx + v$. The idea is that K will be picked based on some performance criteria, not to just to be placed exactly at some a-priori determined locations.

Remark: LQ method can be used for approximates pole assignment (see later chapter).

Example:

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We see from Fig. 4.4 that the nominal system is unstable, and also the system is uncontrollable by open loop step inputs. Let $B = [b_1 \ \ b_2]$, then we see that (A, b_1) is not controllable. We choose

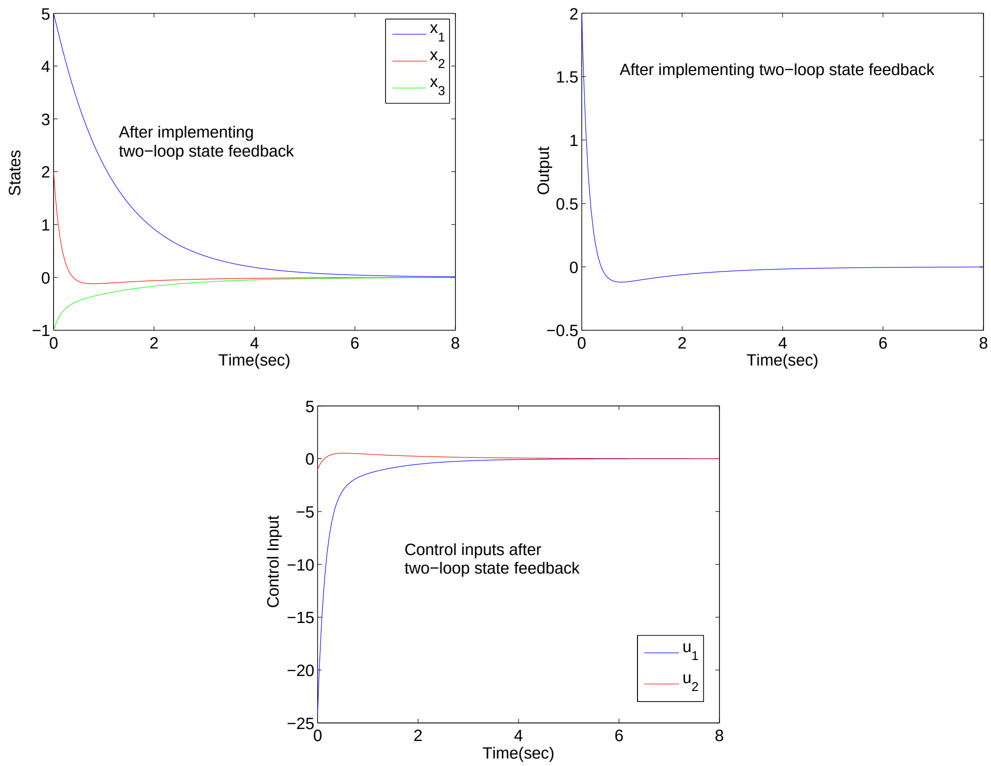

Figure 4.5: Closed loop system after two-loop state feedback

F such that $(A - BF, b_1)$ is controllable.

$$F = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad\qquad g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = Bg$$

For the inner loop,

$$u = -Fx + gv$$

and for the outer loop,

$$v = -Kx + g_1 v_1$$

where g_1 is a parameter to be selected to account for the DC gain. The target closed loop pole locations are $\begin{pmatrix} -3 & -2 & -1 \end{pmatrix}$. As usual, A_1 is transformed into controllable canonical form, the T matrix is computed, and the gain vector K_z is computed as

$$K_z = \begin{pmatrix} 7 & 16 & 9 \end{pmatrix}$$

which is transformed into original system coordinates

$$K = K_z T^{-1} = \begin{pmatrix} 0 & 9 & -2 \end{pmatrix}$$

The regulation problem is shown here. Hence, $v_1 = 0$. See Fig. 4.5.

4.5 State feedback for time varying system

The pole placement technique is appropriate only for linear time invariant systems. How about linear time varying systems, such as obtained by linearizing a nonlinear system about a trajectory?

- Use least norm control
- Make use of *uniform controllability*

Consider the modified controllability (to zero) grammian function:

$$H_\alpha(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)e^{-4\alpha(\tau - t_0)}d\tau.$$

Note that for $\alpha = 0$, $H_\alpha(t_0, t_1) = W_{c,[t_0,t_1]}$, the controllability (to zero) grammian.

Theorem *If $A(\cdot)$ and $B(\cdot)$ are piecewise continuous and if there exists $T > 0$, $h_M \geq h_m > 0$ s.t. for all $t \geq 0$,*

$$0 < h_m I \leq H_0(t, t+T) \leq h_M I$$

then for any $\alpha > 0$, the linear state feedback,

$$u(t) = -F(t)x(t) = -B^T(t)H_\alpha(t, t+T)^{-1}x(t)$$

will result in a closed loop system:

$$\dot{x} = (A(t) - B(t)F(t))x(t)$$

such that for all $x(t_0)$, and all t_0,

$$\|x(t)e^{\alpha t}\| \to 0$$

Example: Consider a subsystem having vibrational modes, which need to be damped out,

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= Cx \end{aligned}$$

$$A(t) = \begin{pmatrix} 0 & -\omega(t) \\ \omega(t) & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 2 + 0.5\sin(0.5t) \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

where $\omega(t) = 2 - \cos(t)$, and state transition matrix

$$\Phi(t, 0) = \begin{pmatrix} \cos(\theta(t)) & \sin(\theta(t)) \\ -\sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix}$$

where $\theta(t) = \int_0^t \omega(t)dt$.

$$H_\alpha(t, t+4\pi) = \int_t^{t+4\pi} \Phi(t, \tau)B(\tau)B^T(\tau)\Phi^T(t, \tau)e^{-4\alpha(\tau - t_0)}d\tau$$

where $T = 4\pi$ has been chosen so that it is a multiple of the fundamental period of the sinusoidal component of $B(t)$.

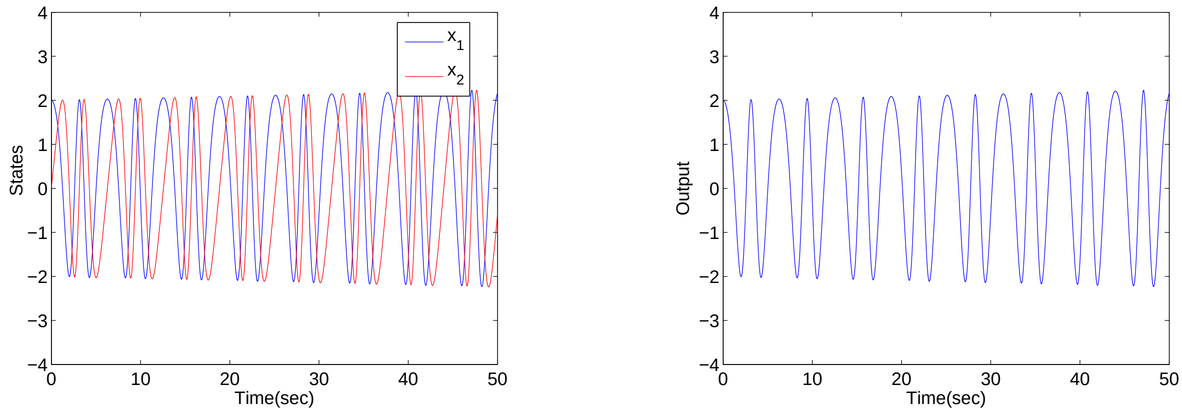

Figure 4.6: Nominal system state (left) and output (right)

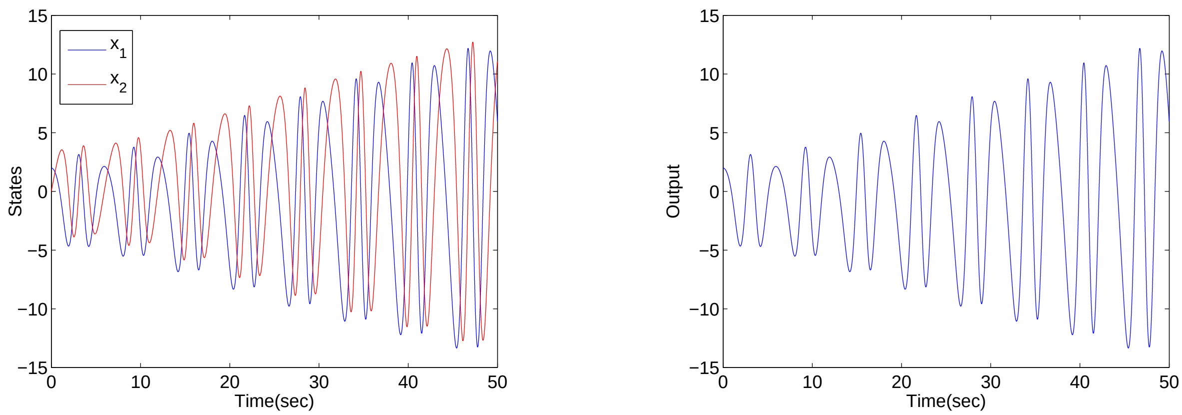

Figure 4.7: Nominal system state (left) and output (right) with open loop control

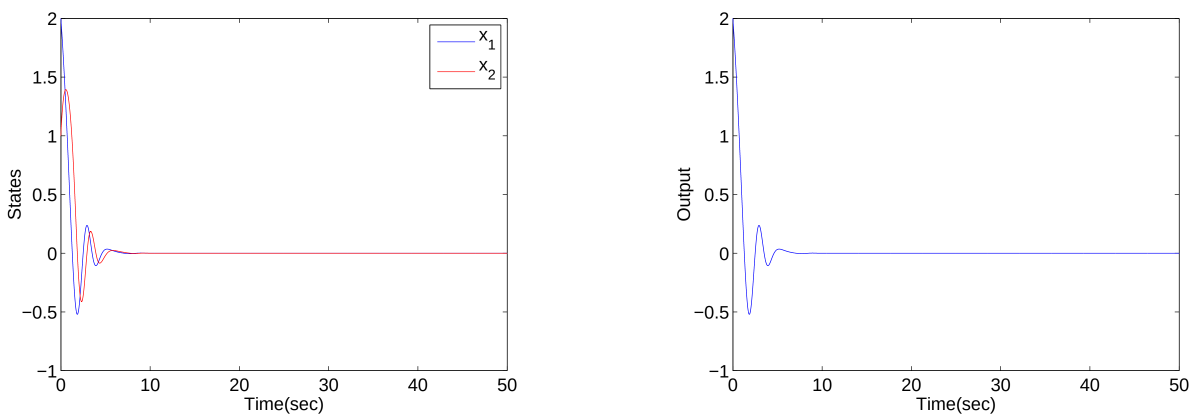

Figure 4.8: Closed loop system state (left) and output (right) for $\alpha = 0.2$

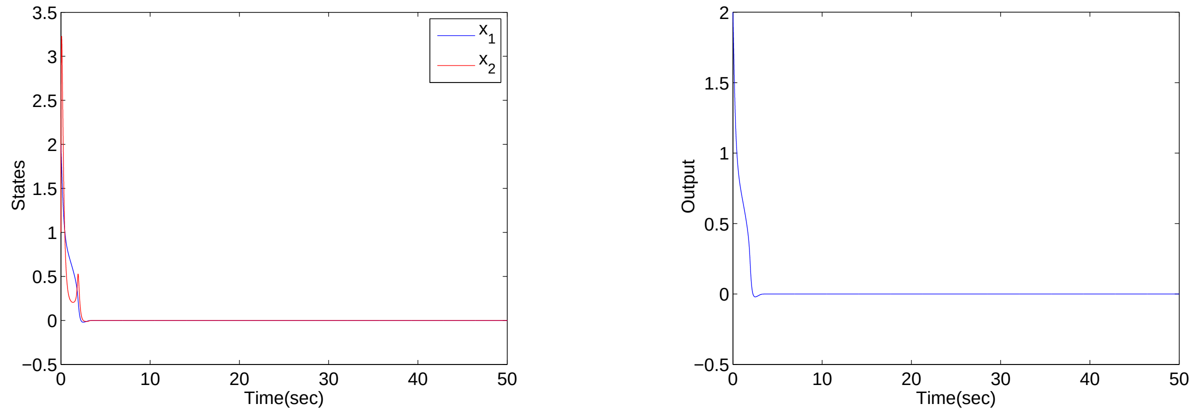

Figure 4.9: Closed loop system state (left) and output (right) for $\alpha = 1.8$

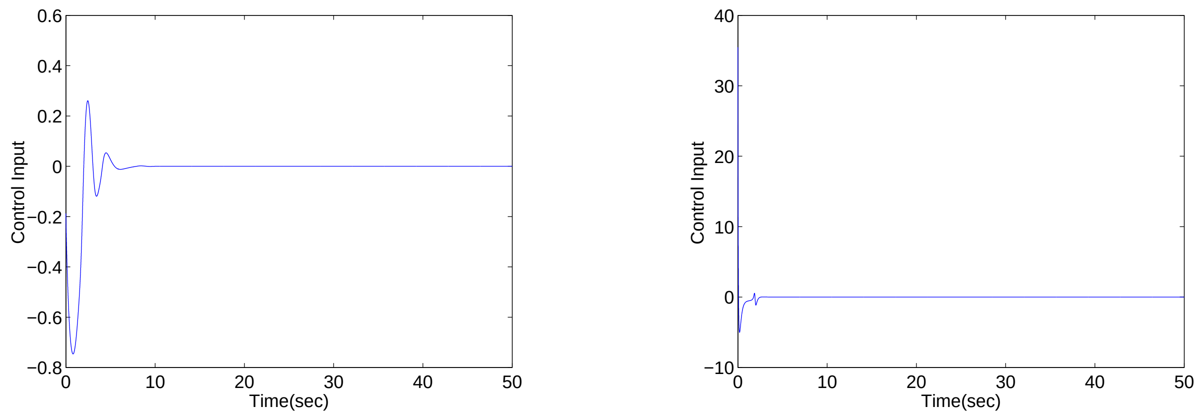

Figure 4.10: Control input for $\alpha = 0.2$ (left) and $\alpha = 1.8$ (right)

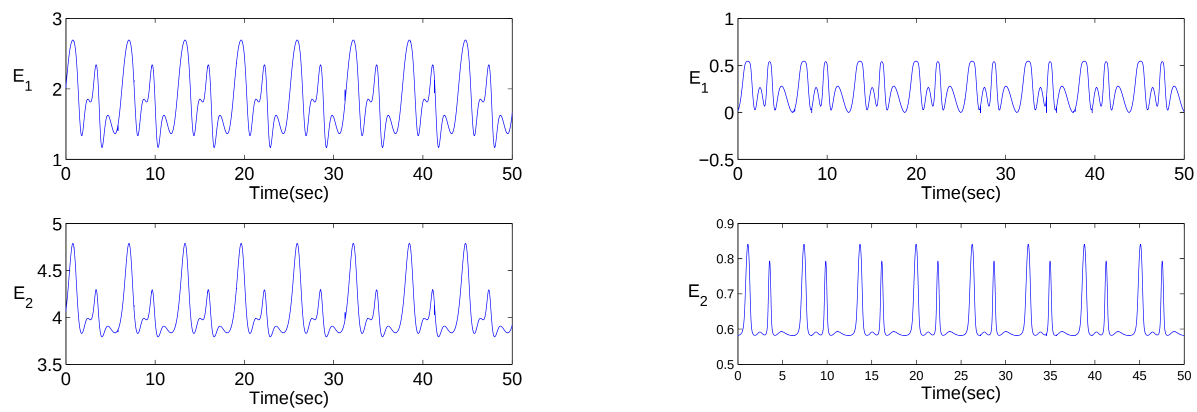

Figure 4.11: Eigenvalues of H_α for $\alpha = 0.2$ (left) and $\alpha = 1.8$ (right)

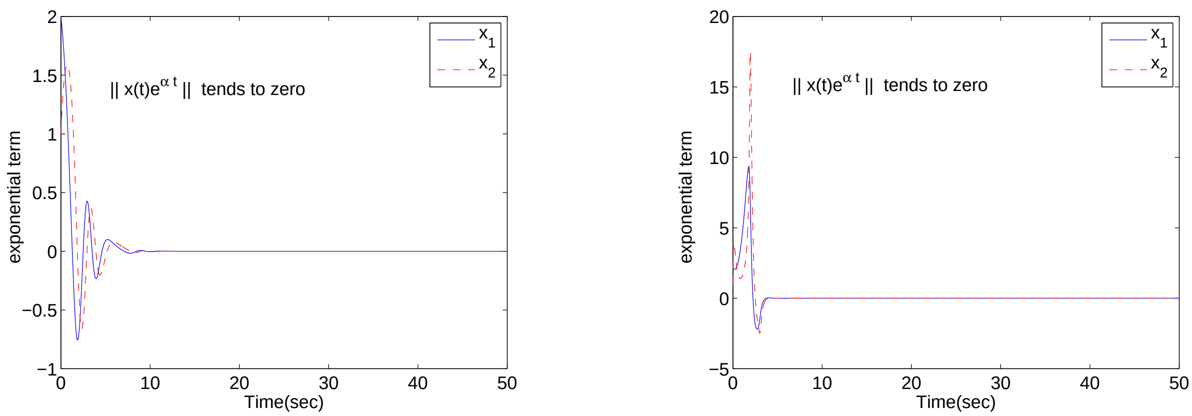

Figure 4.12: $||x(t)e^{\alpha t}||$ for $\alpha = 0.2$ (left) and $\alpha = 1.8$ (right)

Use MATLAB's `quad` to compute the time-varying $H_\alpha(t, t+4\pi)$. Here, the simulation is carried out for $\alpha = 0.2, 1.8$. Fig. 4.6 shows the unforced system, while Fig. 4.7 shows the nominal system subject to unit step input. Figures 4.8 - 4.10 show the response of the closed loop state feedback system. It is clear that the nominal system is unstable by itself, and the state feedback stabilizes the closed loop system.

Remarks:

- H_α can be computed beforehand.
- Can be applied to periodic systems, e.g. swimming machine.
- α is used to choose the decay rate.
- The controllability to 0 map on the interval (t_0, t_1) $L_{c,[t_0,t_1]}$ is:

$$u(\cdot) \mapsto L_{c,[t_0,t_1]}[u(\cdot)] := -\int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)u(\tau)d\tau$$

The least norm solution that steers a state $x(t_0)$ to $x(t_1) = 0$ with respect to the cost function:

$$J_{[t_0,t_1]} = \int_{t_0}^{t_1} u^T(\tau)u(\tau)exp(4\alpha(\tau - t_0))d\tau$$

is:

$$u(\tau) = -e^{-4\alpha(\tau - t_0)}B^T(\tau)\Phi(t_0,\tau)^T H_\alpha(t_0,t_1)^{-1}x(t_0).$$

and when evaluated at $\tau = t_0$,

$$u(t_0) = -B^T(t_0)H_\alpha(t_0,t_1)^{-1}x(t_0).$$

Thus, the proposed control law is the least norm control evaluated at $\tau = t_0$. By relating this to a moving horizon $[t_0, t_1] = [t, t+T]$, where t continuously increases, the proposed control law is the moving horizon version of the least norm control. This avoids the difficulty of receding horizon control where the control gain can become infinite when $t \to t_f$.

- Proof is based on a Lyapunov analysis typical of nonlinear control, and can be found in [Desoer and Callier, 1990, p. 231]

4.6 Observer Design

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The observer problem is that given $y(t)$ and $u(t)$ can we determine the state $x(t)$?

Openloop observer

Suppose that A has eigenvalues on the LHP (stable). Then an open loop obsever is a simulation:

$$\dot{\hat{x}} = A\hat{x} + Bu$$

The observer error dynamics for $e = x - \hat{x}$ are:

$$\dot{e} = Ae$$

Since A is stable, $e \to 0$ exponentially.

The problem with open loop observer is that they do not make use of the output $y(t)$, and also it will not work in the presence of disturbances or if A is unstable.

4.7 Closed loop observer by output injection

Luenberger Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

This looks like the open loop observer except for the last term. Notice that $C\hat{x} - y$ is the output prediction error, also known as the innovation. L is the observer gain.

Let us analyse the error dynamics $e = x - \hat{x}$. Subtracting the observer dynamics by the plant dynamics, and using the fact that $y = Cx$,

$$\dot{e} = Ae + LC(x - \hat{x}) = (A - LC)e.$$

If $A - LC$ is stable (has all its eigenvalues in the LHP), then $e \to 0$ exponentially.

Design of observer gain L:

We use eigenvalue assignment technique to choose L. i.e. choose L so that the eigenvalues of $A - LC$ are at the desired location, $p_1, p_2, \ldots, p_n$.

Fact: *Let $F \in \Re^{n\times n}$. Then, $det(F) = det(F^T)$*

Therefore,

$$det(\lambda I - \bar{F}) = det(\lambda I - \bar{F}^T).$$

Hence, F and F^T have the same eigenvalues. So choosing L to assign the eignvalues of $A - LC$ is the same as choosing L to assign the eigenvalues of

$$(A - LC)^T = A^T - C^T L^T$$

We know how to do this, since this is the state feedback problem for:

$$\dot{x} = A^T x + C^T u, \qquad u = v - L^T C.$$

The condition in which the eigenvalues can be placed arbitrarily is that (A^T, C^T) is controllable. However, from the PBH test, it is clear that:

$$rank\begin{pmatrix}\lambda I - A^T & C^T\end{pmatrix} = rank\begin{pmatrix}\lambda I - A \\ C\end{pmatrix}$$

the LHS is the controllability test, and the RHS is the observability test. Thus, the observer eigenvalues can be placed arbitrarily iff (A, C) is observable.

Example: Consider a second-order system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$$

where $u = 3 + 0.5\sin(0.75t)$ is the control input, and $\zeta = 1, \omega_n = 1rad/s$. The state space representation of the system is

$$\frac{d}{dt}\begin{pmatrix}x \\ \dot{x}\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -1 & -2\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix}u$$
$$y = \begin{pmatrix}0 & 1\end{pmatrix}x$$

The observer is represented as:

$$\frac{d}{dt}\begin{pmatrix}\hat{x} \\ \dot{\hat{x}}\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -1 & -2\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix}u - LC(\hat{x} - x)$$

The observer gain L is computed by placing the observer poles at $(-0.25 \quad -0.5)$.

$$L = \begin{pmatrix}0.875 \\ -1.25\end{pmatrix}$$

See Fig. 4.13 for the plots of the observer states following and catching up with the actual system states.

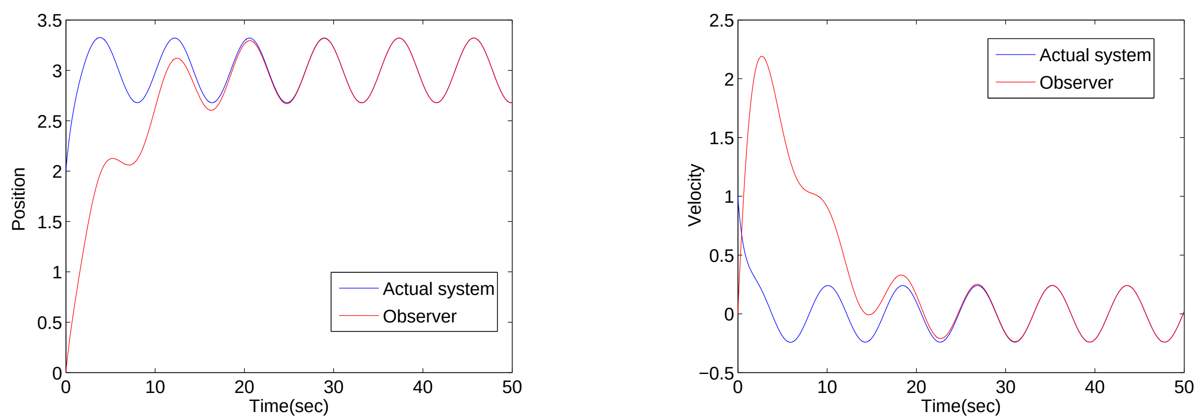

Figure 4.13: Observer with output injection

4.8 Properties of observer

- The observer is unbiased. The transfer function from u to $\hat{x}$ is the same as the transfer function from u to x.

- The observer error $e = \hat{x} - x$ is uncontrollable from the control u. This is because,

$$\dot{e} = (A - LC)e$$

  no matter what the control u is.

- Let $\nu(t) := y(t) - C\hat{x}(t)$ be the innovation. Since $\nu(t) = Ce(t)$, the transfer function from $u(t)$ to $\nu(t)$ is 0.

- This has the significance that feedback control of the innovation of the form

$$U(s) = -K\hat{X}(s) - Q(s)\nu(s)$$

  where $A - BK$ is stable, and $Q(s)$ is *any* stable controller (i.e. $Q(s)$ itself does not have any unstable poles), is necessarily stable. In fact, any stabilizing controller is of this form! (see Goodwin et al. Section 18.6 for proof of necessity)

- Transfer function of the observer:

$$\hat{X}(s) = T_1(s)U(s) + T_2(s)Y(s)$$

  where

$$T_1(s) := (sI - A + LC)^{-1}B$$
$$T_2(s) := (sI - A + LC)^{-1}L$$

  Both $T_1(s)$ and $T_2(s)$ have the same denominator, which is the characteristic polynomial of the observer dynamics, $\Psi_{obs}(s) = det(sI - A + LC)$.

- With $Y(s)$ given by $G_o(s)U(s)$ where $G_o(s) = C(sI - A)^{-1}B$ is the open loop plant model, the transfer function from $u(t)$ to $\hat{X}(t)$ is

$$\begin{aligned}\hat{X}(s) &= [T_1(s) + T_2(s)G_0(s)]\,U(s)\\ &= (sI - A)^{-1}BU(s)\end{aligned}$$

  i.e. the same as the open loop transfer function, from $u(t)$ to $x(t)$. In this sense, the observer is unbiased.

Where should the observer poles be ?

Theoretically, the observer error will decrease faster if the eigenvalues of the $A-LC$ are further to the left (more negative). However, effects of measurement noise can be filtered out if eigenvalues are slower. A rule of thumb is that if noise bandwidth is $Brad/s$, the fastest eigenvalue should be greater than $-B$ (i.e. slower than the noise band) (Fig. 4.14). This way, observer acts as a filter.

If observer states are used for state feedback, then the slowest eigenvalues of $A - LC$ should be faster than the eigenvalue of the state feedback system $A - BK$.

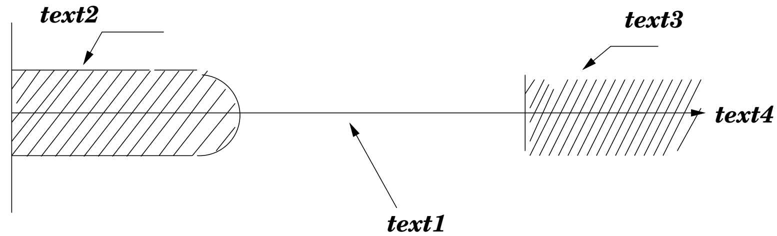

Figure 4.14: Placing observer poles

4.9 Observer state feedback

For a system

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{4.2}$$

the observer structure is

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{4.3}$$

where $y(t) - C\hat{x}(t) =: \nu(t)$ is called innovation, which will be discussed later. Define observer error

$$e(t) = \hat{x}(t) - x(t)$$

Stacking up (4.2) and (4.3) in matrix form,

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ LC & A - LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} u$$

Transforming the coordinates from $\begin{pmatrix} x \\ \hat{x} \end{pmatrix}$ to $\begin{pmatrix} e \\ \hat{x} \end{pmatrix}$,

$$\begin{pmatrix} \dot{e} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A - LC & 0 \\ -LC & A \end{pmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u$$

As seen from Fig. 4.15, there is no effect of control u on the error e, and if the gain vector L is designed (by separation principle, see below) such that the roots of characteristic polynomial

$$\psi_{A-LC}(s) = det(sI - A + LC)$$

are in the LHP, then the error tends asymptotically to zero. This means the observer has estimated the system states to some acceptable level of accuracy. Now, the gain vector K can be designed to implement feedback using the observer state:

$$u = v - K\hat{x}$$

where v is an exogenous control.

- **Separation principle** - the set of eigenvalues of the complete system is the union of the eigenvalues of the state feedback system and the eigenvalues of the observer system. Hence, state feedback and observer can in principle be designed separately.

$$eigenvalues = eig(A - BK) \cup eig(A - LC)$$

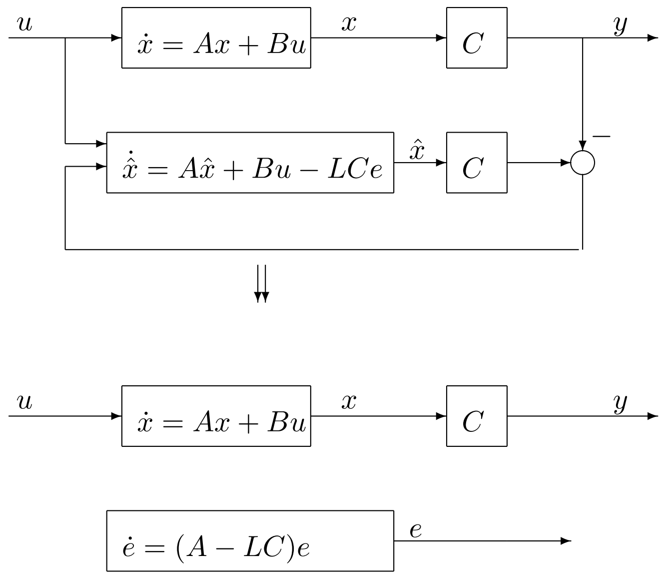

Figure 4.15: Observer coordinate transformation

- Using observer state feedback, the transfer function from v to x is the same as in state feedback system:

$$X(s) = (sI - (A - BK))^{-1} BV(s)$$

4.9.1 Innovation Feedback

As seen earlier, the term "innovation" is the error in output estimation, and is defined by

$$\nu(t) := (y - C\hat{x}) \tag{4.4}$$

To see how the innovation process is used in observer state feedback, let us first look at the transfer function form of the state feedback law. From (4.3), we can write

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$

Taking Laplace transform, we get

$$\begin{aligned}
\hat{X}(s) &= (sI - A + LC)^{-1} BU(s) + (sI - A + LC)^{-1} LY(s) \\
&= \frac{adj(sI - A + LC)B}{det(sI - A + LC)} U(s) + \frac{adj(sI - A + LC)L}{det(sI - A + LC)} Y(s) \\
&= T_1(s)U(s) + T_2(s)Y(s)
\end{aligned}$$

The state feedback law is

$$u(t) = -K\hat{x}(t) + v(t) \tag{4.5}$$

Taking Laplace transform,

$$
\begin{aligned}
U(s) &= -K\hat{X}(s) + V(s) \\
&= -K(T_1(s)U(s) + T_2(s)Y(s)) \\
(1 + KT_1(s))U(s) &= -KT_2(s) + V(s) \\
\frac{L(s)}{E(s)}U(s) &= -\frac{P(s)}{E(s)}Y(s) + V(s)
\end{aligned}
$$

where

$$
\begin{aligned}
E(s) &= det(sI - A + LC) \\
L(s) &= det(sI - A + LC + BK) \\
P(s) &= Kadj(sI - A)L
\end{aligned}
$$

The expression for $P(s)$ has been simplified using a matrix inversion lemma, see Goodwin, p. 522.

The nominal plant TF is

$$
G_0(s) = \frac{Cadj(sI - A)B}{det(sI - A)} = \frac{B_0(s)}{A_0(s)}
$$

The closed loop transfer function from $V(s)$ to $Y(s)$ is

$$
\begin{aligned}
\frac{Y(s)}{V(s)} &= \frac{B_0(s)E(s)}{A_0(s)L(s) + B_0(s)P(s)} \\
&= \frac{B_0(s)}{det(sI - A + BK)} \\
&= \frac{B_0(s)}{F(s)}
\end{aligned}
$$

From (4.4), we have

$$
\begin{aligned}
\nu(s) &= Y(s) - C\hat{X}(s) \\
&= Y(s) - C(T_1(s)U(s) + T_2(s)Y(s)) \\
&= (1 - CT_2(s))Y(s) - CT_1(s)U(s)
\end{aligned}
$$

It can be shown (see Goodwin, p. 537) that

$$
\begin{aligned}
1 - CT_2(s) &= \frac{A_0(s)}{E(s)} \\
CT_1(s) &= \frac{B_0(s)}{E(s)}
\end{aligned}
$$

Therefore

$$
\nu(s) = \frac{A_0(s)}{E(s)}Y(s) - \frac{B_0(s)}{E(s)}U(s)
$$

Augmenting (4.5) with innovation process (see Fig. 4.16),

$$
u(t) = v(t) - K\hat{x}(t) + Q_u(s)\nu(s)
$$

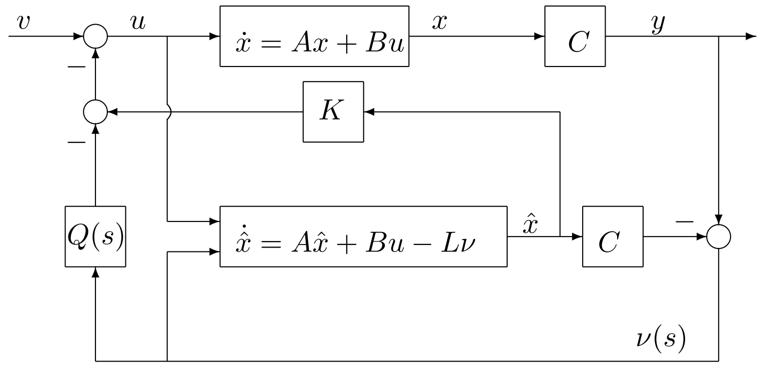

Figure 4.16: Observer state feedback augmented with innovation feedback

where $Q(s)$ is any stable transfer function (filter)

Then

$$\frac{L(s)}{E(s)}U(s) = V(s) - \frac{P(s)}{E(s)}Y(s) + Q_u(s)\left(\frac{A_0(s)}{E(s)}Y(s) - \frac{B_0(s)}{E(s)}U(s)\right)$$

The nominal sensitivity functions which define the robustness and performance criteria are modified affinely by $Q_u(s)$:

$$S_0(s) = \frac{A_0(s)L(s)}{E(s)F(s)} - Q_u(s)\frac{B_0(s)A_0(s)}{E(s)F(s)}$$
$$T_0(s) = \frac{B_0(s)P(s)}{E(s)F(s)} + Q_u(s)\frac{B_0(s)A_0(s)}{E(s)F(s)}$$

For plants the are open loop stable with tolerable pole locations, we can set $K - 0$ so that

$$F(s) = A_0(s)$$
$$L(s) = E(s)$$
$$P(s) = 0$$

so that

$$S_0(s) = 1 - Q_u(s)\frac{B_0(s)}{E(s)}$$
$$T_0(s) = Q_u(s)\frac{B_0(s)}{E(s)}$$

In this case, it is common to use $Q(s) := Q_u(s)\frac{A_0(s)}{E(s)}$. Then

$$S_0(s) = 1 - Q(s)G_0(s)$$
$$T_0(s) = Q_u(s)G_0(s)$$

Thus the design of $Q_u(s)$(or $Q(s)$) can be used to directly influence the sensitivity functions.

Example: Consider a previous example:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$$

where $u = 3 + 0.5\sin(0.75t)$ is the control input, and $\zeta = 1, \omega_n = 1rad/s$. The state space representation of the system is

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

The observer is represented as:

$$\frac{d}{dt}\begin{pmatrix} \hat{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u - LC(\hat{x} - x)$$

The observer gain L is computed by placing the observer poles at $(-0.25 \quad -0.5)$.

$$L = \begin{pmatrix} 0.875 \\ -1.25 \end{pmatrix}$$

The state feedback law is

$$u = v - K\hat{x} + Q(s)\nu(s)$$

where $K = (1 \ \ 1)$ is computed by placing closed loop poles at $(-1 \quad -2)$. The relevant transfer functions as seen above are:

$$\begin{aligned}
G_0(s) &= \frac{B_0(s)}{A_0(s)} = \frac{1}{s^2 + 2s + 1} \\
E(s) &= det(sI - A + LC) = s^2 + 0.75s + 0.125 \\
L(s) &= det(sI - A + LC + BK) = s^2 + 2.375s - 3.1562 \\
P(s) &= Kadj(sI - A)L = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} 0.875 \\ -1.25 \end{pmatrix} = -0.375(s+1) \\
F(s) &= det(sI - A + BK) = s^2 + 3.625s - 0.25
\end{aligned}$$

The sensitivity functions are

$$S_0(s) = \frac{s^4 + 4.375s^3 + 2.594s^2 - 3.937s - 3.156}{s^4 + 4.375s^3 + 2.594s^2 + 0.2656s - 0.03125} - Q_u(s)\frac{s^2 + 2s + 1}{s^4 + 4.375s^3 + 2.594s^2 + 0.2656s - 0.03125}$$
$$T_0(s) = \frac{-0.375s - 0.375}{s^4 + 4.375s^3 + 2.594s^2 + 0.2656s - 0.03125} + Q_u(s)\frac{s^2 + 2s + 1}{s^4 + 4.375s^3 + 2.594s^2 + 0.2656s - 0.03125}$$

Now, $Q_u(s)$ can be selected to reflect the robustness and performance requirements of the system.

4.10 Internal model principle in states space

Method 1 Disturbance estimate

Suppose that disturbance enters a state space system:

$$\dot{x} = Ax + B(u + d)$$
$$y = Cx$$

Assume that disturbance $d(t)$ is unknown, but we know that it satisfies some differential equations. This implies that $d(t)$ is generated by an exo-system.

$$\dot{x}_d = A_d x_d$$
$$d = C_d x_d$$

Since,

$$D(s) = C_d(sI - A_d)^{-1} x_d(0) = C_d \frac{Adj(sI - A_d)}{det(sI - A_d)} x_d(0)$$

where $x_d(0)$ is iniital value of $x_d(t = 0)$. Thus, the disturbance generating polynomial is nothing but the characteristic polynomial of A_d,

$$\Gamma_d(s) = det(sI - A_d)$$

For example, if $d(t)$ is a sinusoidal signal,

$$\begin{pmatrix} \dot{x}_{d1} \\ \dot{x}_{d2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_{d1} \\ x_{d2} \end{pmatrix}$$
$$d = x_{d1}$$

The characteristic polynomial, as expected, is:

$$\Gamma_d(s) = det(sI - A_d) = s^2 + \omega^2$$

If we *knew* $d(t)$ then an obvious control is:

$$u = -d + v - Kx$$

where K is the state feedback gain. However, $d(t)$ is generally unknown. Thus, we estimate it using an observer. First, augment the plant model.

$$\begin{pmatrix} \dot{x} \\ \dot{x}_d \end{pmatrix} = \begin{pmatrix} A & BC_d \\ 0 & A_d \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix}$$

Notice that the augmented system is not controllable from u. Nevertheless, if d has effect on y, it is observable from y.

Thus, we can design an observer for the augmented system, and use the observer state for feedback:

$$\frac{d}{dt} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix} = \begin{pmatrix} A & BC_d \\ 0 & A_d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \left\{ y - \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix} \right\}$$
$$u = -C_d \hat{x}_d + v - K\hat{x} = v - \begin{pmatrix} K & C_d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix}$$

where $L = [L_1^T, L_2^T]^T$ is the observer gain. The controller can be simplified to be:

$$\frac{d}{dt}\begin{pmatrix}\hat{x}\\ x_d\end{pmatrix} = \begin{pmatrix}A - BK - L_1 C & 0\\ -L_2 C & A_d\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{x}_d\end{pmatrix} - \begin{pmatrix}-B & L_1\\ 0 & L_2\end{pmatrix}\begin{pmatrix}v\\ y\end{pmatrix}$$

$$u = -\begin{pmatrix}K & C_d\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{x}_d\end{pmatrix} + v$$

The $y(t) \to u(t)$ controller transfer function $G_{yu}(s)$ has, as poles, eigenvalues of $A - BK - L_1 C$ and of A_d. Since $\Gamma_d(s) = det(sI - A_d)$ the disturbance generating polynomial, the controller has $\Gamma_d(s)$ in its denominator.

This is exactly the **Internal Model Principle**.
Example: For the system

$$\dot{x} = \begin{pmatrix}0 & 1\\ -1 & -2\end{pmatrix}x + \begin{pmatrix}0\\ 1\end{pmatrix}u$$
$$y = \begin{pmatrix}0 & 1\end{pmatrix}$$

The disturbance is a sinusoidal signal $d(t) = \sin(0.5t)$. The reference input is a constant step $v(t) = 3$. The gains were designed to be:

$$K = \begin{pmatrix}5 & 3\end{pmatrix}$$

with closed loop poles at $(-2 \quad -3)$.

$$L = \begin{pmatrix}L_1^T & L_2^T\end{pmatrix}^T = \begin{pmatrix}2.2 & 0.44 & 2.498 & -0.746\end{pmatrix}^T$$

with observer poles at $(-0.6 \quad -0.5 \quad -1.7 \quad -1.4)$. Figures 4.17 - 4.19 show the simulation results. For a reference signal $v(t) = 3 + 0.5\sin 0.75t$, the results are shown in Figures 4.20 - 4.22. In this case,

$$K = \begin{pmatrix}0.75 & 2\end{pmatrix}$$

with closed loop poles at $(-0.5 \quad -3.5)$.

$$L = \begin{pmatrix}L_1^T & L_2^T\end{pmatrix}^T = \begin{pmatrix}-13.2272 & 6.7 & 8.2828 & 14.493\end{pmatrix}^T$$

with observer poles at $(-0.6 \quad -0.5 \quad -1.7 \quad -1.4)$.

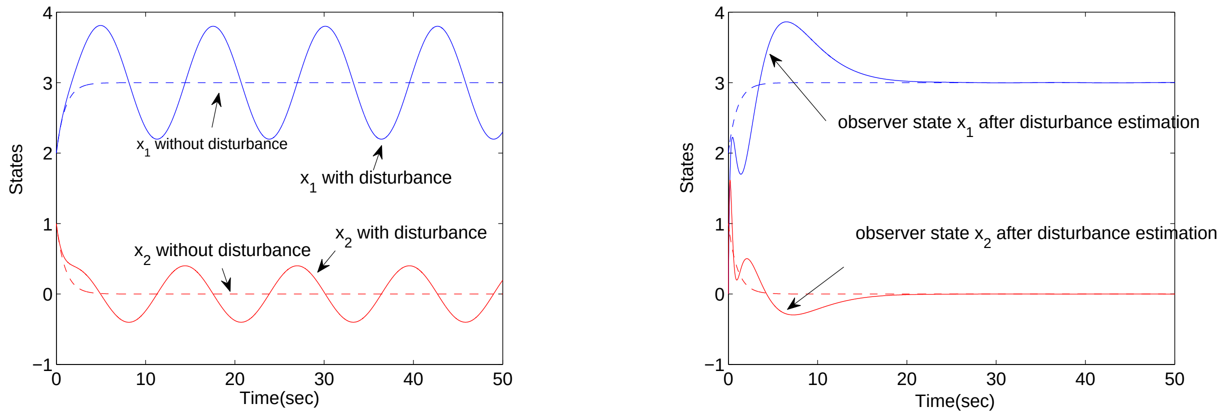

Figure 4.17: States of (left)open loop system and (right)observer state feedback

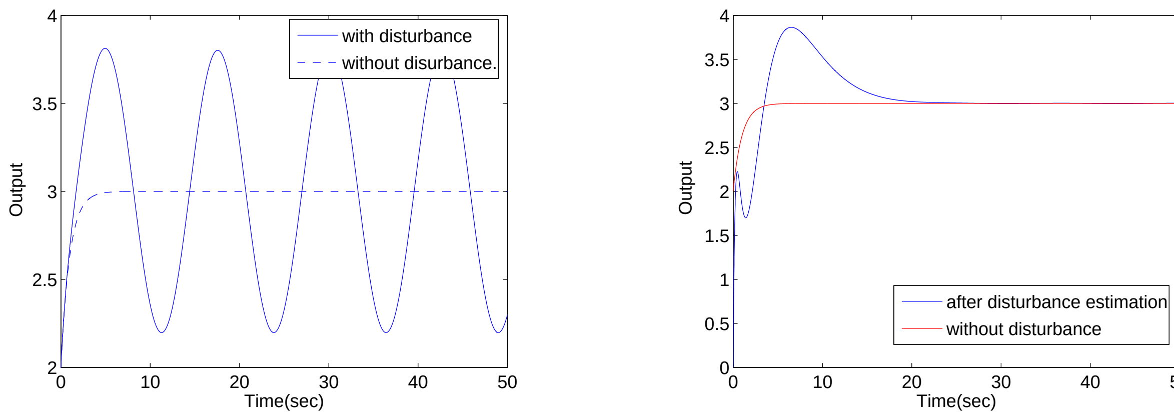

Figure 4.18: Output of (left)open loop system and (right)observer state feedback

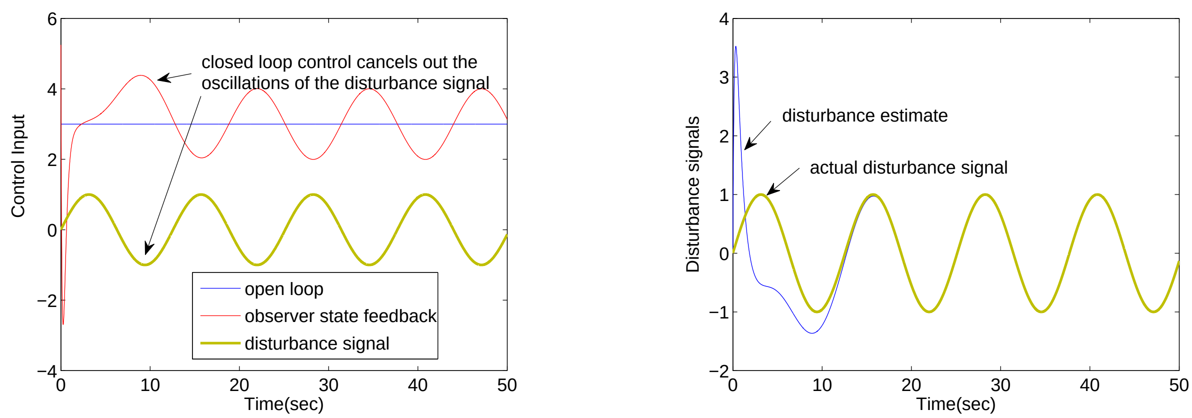

Figure 4.19: (Left)Control and disturbance signals; (right)disturbance estimation

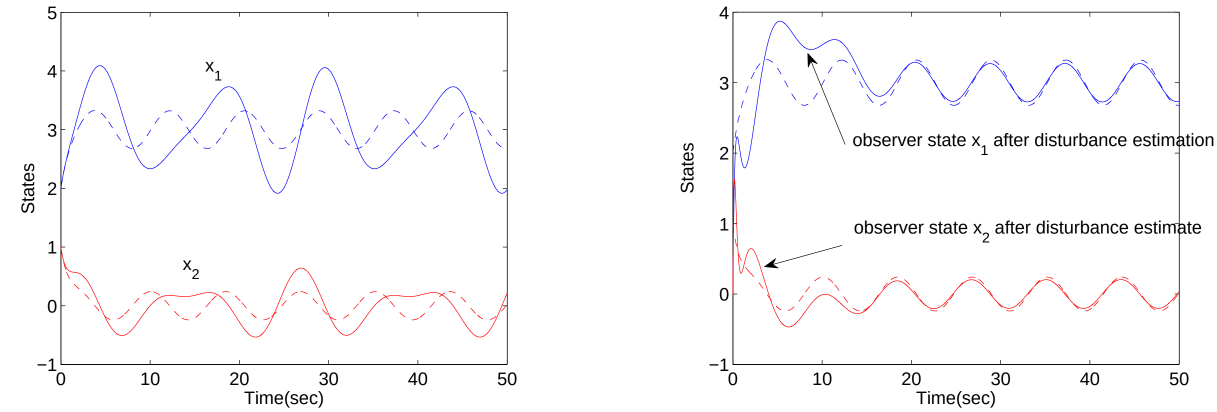

Figure 4.20: States of (left)open loop system and (right)observer state feedback

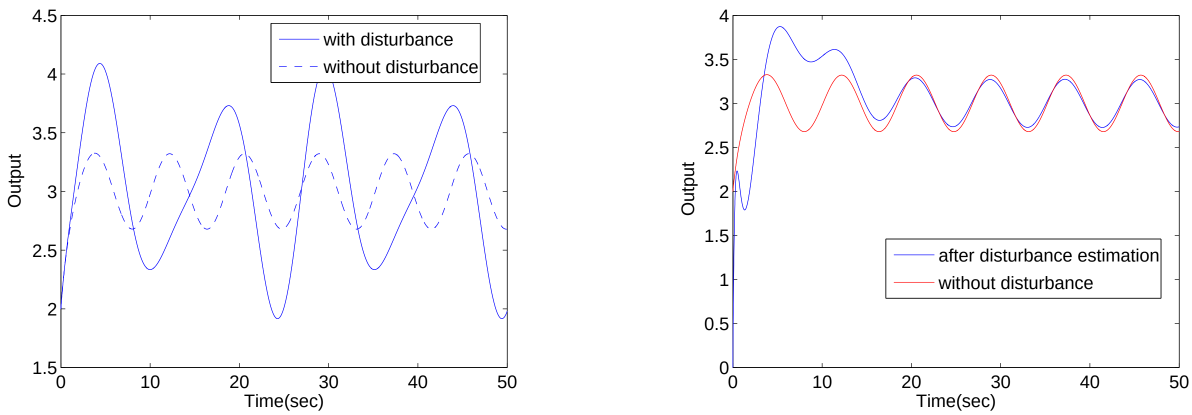

Figure 4.21: Output of (left)open loop system and (right)observer state feedback

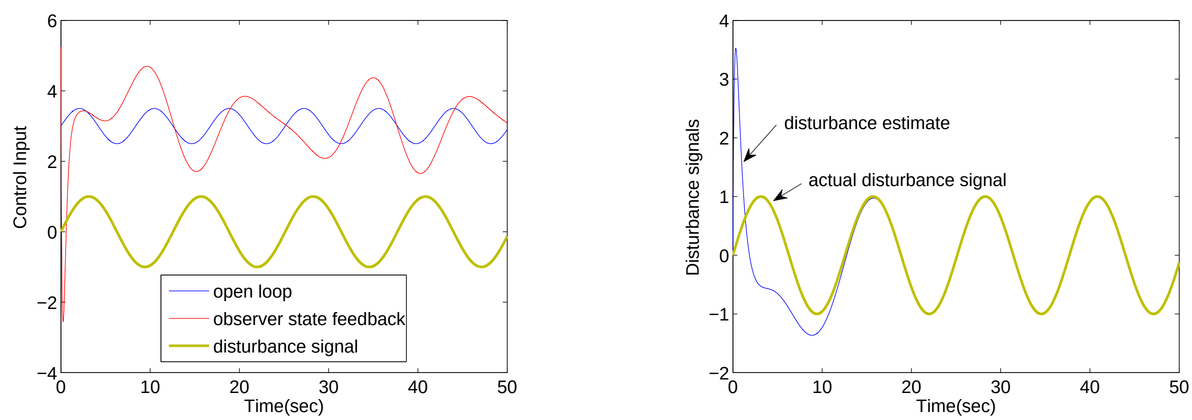

Figure 4.22: (Left)Control and disturbance signals; (right)disturbance estimation

Method 2: Augmenting plant dynamics

In this case, the goal is to introduce the disturbance generating polynomial into the controller dynamics by filtering the output $y(t)$. Let $\dot{x}_d = A_d x_d$, $d = C_d x_d$ be the disturbance model.

Nominal plant and output filter:

$$\begin{aligned}\dot{x} &= Ax + Bu + Bd \\ y &= Cx \\ \dot{x}_a &= A_d^T x_a + C_d^T y(t)\end{aligned}$$

Stabilize the augmented system using (observer) state feedback:

$$u = -[K_o \quad K_a]\begin{pmatrix}\hat{x} \\ x_a\end{pmatrix}$$

where $\hat{x}$ is the observer estimate of the original plant itself.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).$$

Notice that x_a need not be estimated since it is generated by the controller itself!

The transfer function of the controller is: $C(s) =$

$$\begin{pmatrix}K_o & K_a\end{pmatrix}\begin{pmatrix}sI - A + BK_o + LC & BK_o \\ 0 & sI - A_d\end{pmatrix}^{-1}\begin{pmatrix}L \\ C_d^T\end{pmatrix}$$

from which it is clear that the its denominator has $\Gamma_d(s) = det(sI - A_d)$ in it. i.e. the Internal Model Principle.

An **intuitive** way of understanding this approach:

For concreteness, assume that the disturbance $d(t)$ is a sinusoid with frequency ω.

- Suppose that the closed loop system is stable. This means that for any bounded input, any internal signals will also be bounded.
- For the sake of contradiction, if some residual sinusoidal response in $y(t)$ still remains:

$$Y(s) = \frac{\alpha(s,0)}{s^2 + \omega^2}$$

- The augmented state is the filtered version of $Y(s)$,

$$U(s) = -K_a X_a(s) = \frac{K_a \alpha(s,0)}{s^2 + \omega^2}$$

  The time response of $x_a(t)$ is of the form

$$x_a(t) = \gamma sin(\omega t + \phi_1) + \delta \cdot t \cdot sin(\omega t + \phi_2)$$

  The second term will be unbounded.

- Since $d(t)$ is a bounded sinusoidal signal, $x_a(t)$ must also be bounded. This must mean that $y(t)$ does not contain sinusoidal components with frequency ω.

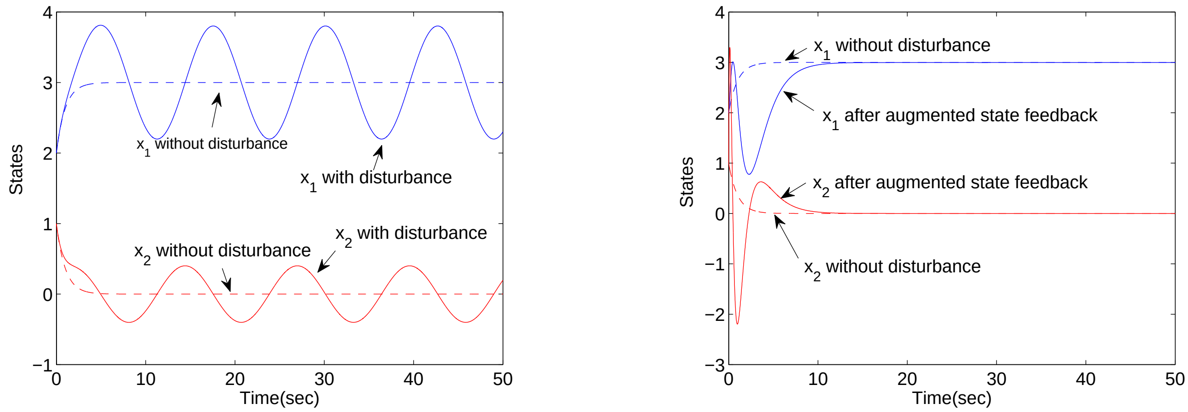

Figure 4.23: States of (left)open loop system and (right)augmented state feedback

The most usual case is to combat constant disturbances using integral control. In this case, the augmented state is:

$$x_a(t) = \int_0^t y(\tau)d\tau.$$

It is clear that if the output converges to some steady value, $y(t) \to y_\infty$, y_∞ must be 0. Or otherwise $x_a(t)$ will be unbounded.

Example: Consider the same example as before. But this time, we do not use an observer to estimate the disturbance. The dynamics is augmented to include a filter. The gains were selected as:

$$L = \begin{pmatrix} 5.7 & 2.3 \end{pmatrix}^T$$

with observer poles at $(-3.5 \quad -4.2)$.

$$K_o = (3.02 \quad 2.18) \qquad K_a = (0.5 \quad 0.76)$$

with the closed loop poles at $(-2.1 \quad -1.2 \quad -2.3 \quad -0.6)$. The results are shown in Figures 4.23 - 4.25 for a constant reference input $v(t) = 3$. The filter state contains the frequency of the disturbance signal, and hence the feedback law contains that frequency. Hence the control signal cancels out the disturbance. This is seen from the output, which does not contain any sinusoidal component.

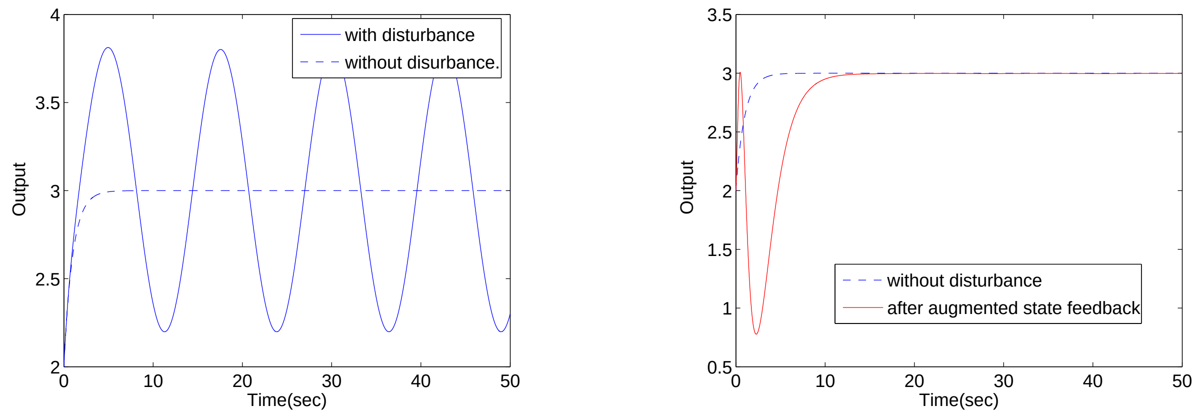

Figure 4.24: Output of (left)open loop system and (right)augmented state feedback

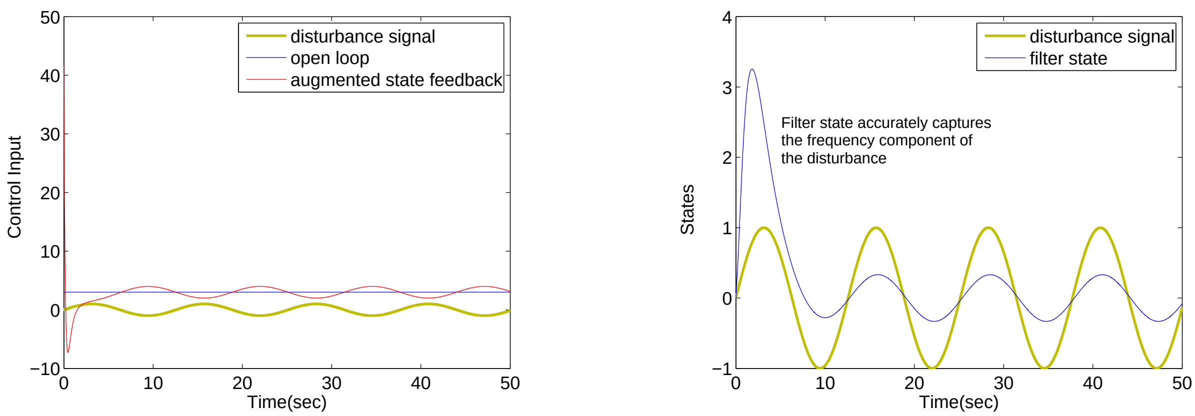

Figure 4.25: (left)control and disturbance signals; (right)filter state x_a