ECE 680 Fall 2013

# Pontryagin's Minimum Principle[1]

In this handout, we provide a derivation of the minimum principle of Pontryagin, which is a generalization of the Euler-Lagrange equations that also includes problems with constraints on the control inputs. Only a special case of the minimum principle is stated. However, this special case covers a large class of control problems. We consider the problem of minimizing the performance index given by

$$J = \Phi(\boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} F(\boldsymbol{x}(t), \boldsymbol{u}(t))dt \tag{1}$$

subject to

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad \text{and} \quad \boldsymbol{x}(t_f) = \boldsymbol{x}_f. \tag{2}$$

We consider two cases: fixed final state and free final state.

To proceed we define the Hamiltonian function $H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})$ as

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = H = F + \boldsymbol{p}^\top \boldsymbol{f},$$

where the costate vector $\boldsymbol{p}$ will be determined in the analysis to follow.

We discuss the case when the final time $t_f$ is fixed. We follow the development of Luenberger [1, Section 11.1]. We "adjoin" to $J$ additional terms that sum up to zero. We note that because the state trajectories must satisfy the equation, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, we have

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \dot{\boldsymbol{x}} = \boldsymbol{0}. \tag{3}$$

We introduce the modified objective performance index,

$$\tilde{J} = J + \int_{t_0}^{t_f} \boldsymbol{p}(t)^\top (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \dot{\boldsymbol{x}})\, dt. \tag{4}$$

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Because for any state trajectory, (3) is satisfied, for any choice of $\boldsymbol{p}(t)$ the value of $\tilde{J}$ is the same as that of $J$. Therefore, we can express $\tilde{J}$ as

$$\begin{aligned}\tilde{J} &= \Phi(\boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} F(\boldsymbol{x}(t), \boldsymbol{u}(t))\, dt + \int_{t_0}^{t_f} \boldsymbol{p}(t)^\top (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \dot{\boldsymbol{x}})\, dt \\ &= \Phi(\boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} (H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) - \boldsymbol{p}^\top \dot{\boldsymbol{x}})\, dt. \qquad (5)\end{aligned}$$

Let $\boldsymbol{u}(t)$ be a nominal control strategy; it determines a corresponding state trajectory $\boldsymbol{x}(t)$. If we apply another control strategy, say $\boldsymbol{v}(t)$, that is "close" to $\boldsymbol{u}(t)$, then $\boldsymbol{v}(t)$ will produce a state trajectory close to the nominal trajectory. This new state trajectory is just a perturbed version of $\boldsymbol{x}(t)$ and it can be represented as

$$\boldsymbol{x}(t) + \delta\boldsymbol{x}(t).$$

The change in the state trajectory yields a corresponding change in the modified performance index. We represent this change as $\delta\tilde{J}$; it has the form,

$$\delta\tilde{J} = \Phi(\boldsymbol{x}(t_f) + \delta\boldsymbol{x}(t_f)) - \Phi(\boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} (H(\boldsymbol{x} + \delta\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) - \boldsymbol{p}^\top \delta\dot{\boldsymbol{x}})\, dt. \qquad (6)$$

We evaluate the integral above. Integrating by parts gives

$$\int_{t_0}^{t_f} \boldsymbol{p}^\top \delta\dot{\boldsymbol{x}}\, dt = \boldsymbol{p}(t_f)^\top \delta\boldsymbol{x}(t_f) - \boldsymbol{p}(t_0)^\top \delta\boldsymbol{x}(t_0) - \int_{t_0}^{t_f} \dot{\boldsymbol{p}}^\top \delta\boldsymbol{x}\, dt. \qquad (7)$$

Note that $\delta\boldsymbol{x}(t_0) = \boldsymbol{0}$ because a change in the control strategy does not change the initial state. Taking into account the above, we represent (6) as

$$\begin{aligned}\delta\tilde{J} &= \Phi(\boldsymbol{x}(t_f) + \delta\boldsymbol{x}(t_f)) - \Phi(\boldsymbol{x}(t_f)) - \boldsymbol{p}(t_f)^\top \delta\boldsymbol{x}(t_f) \\ &\quad + \int_{t_0}^{t_f} (H(\boldsymbol{x} + \delta\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) + \dot{\boldsymbol{p}}^\top \delta\boldsymbol{x})\, dt.\end{aligned}$$

We next replace $(\Phi(\boldsymbol{x}(t_f) + \delta\boldsymbol{x}(t_f)) - \Phi(\boldsymbol{x}(t_f)))$ with its first-order approximation, and add and subtract the term $H(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p})$ under the integral to obtain

$$\begin{aligned}\delta\tilde{J} &= \left(\nabla_{\boldsymbol{x}}\Phi|_{t=t_f} - \boldsymbol{p}(t_f)\right)^\top \delta\boldsymbol{x}(t_f) \\ &\quad + \int_{t_0}^{t_f} (H(\boldsymbol{x} + \delta\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) + H(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) + \dot{\boldsymbol{p}}^\top \delta\boldsymbol{x})\, dt.\end{aligned}$$

Replacing $(H(\boldsymbol{x}+\delta\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})-H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p}))$ with its first-order approximation,

$$H(\boldsymbol{x}+\delta\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})-H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})=\frac{\partial H}{\partial \boldsymbol{x}}\delta\boldsymbol{x},$$

gives

$$\begin{aligned}\delta\tilde{J} &= \left(\nabla_{\boldsymbol{x}}\Phi|_{t=t_f}-\boldsymbol{p}(t_f)\right)^\top \delta\boldsymbol{x}(t_f)\\ &\quad+\int_{t_0}^{t_f}\left(\left(\frac{\partial H}{\partial \boldsymbol{x}}+\dot{\boldsymbol{p}}^\top\right)\delta\boldsymbol{x}+H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})-H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p})\right)dt.\end{aligned}$$

Selecting in the above $\boldsymbol{p}$ as the solution to the differential equation

$$\frac{\partial H}{\partial \boldsymbol{x}}+\dot{\boldsymbol{p}}^\top=\mathbf{0}^\top$$

with the final condition

$$\boldsymbol{p}(t_f)=\nabla_{\boldsymbol{x}}\Phi|_{t=t_f},$$

reduces $\delta\tilde{J}$ to

$$\delta\tilde{J}=\int_{t_0}^{t_f}\left(H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})-H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p})\right)dt.$$

If the original control $\boldsymbol{u}(t)$ is optimal, then we should have $\delta\tilde{J}\geq 0$, that is, $H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})\geq H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p})$. We summarize our development in the following theorem.

**Theorem 1** *Necessary conditions for $\boldsymbol{u}\in U$ to minimize (1) subject to (2) are:*

$$\dot{\boldsymbol{p}}=-\left(\frac{\partial H}{\partial \boldsymbol{x}}\right)^\top,$$

*where $H=H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p})=F(\boldsymbol{x},\boldsymbol{u})+\boldsymbol{p}^\top\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$,*

$$H(\boldsymbol{x}^*,\boldsymbol{u}^*,\boldsymbol{p}^*)=\min_{\boldsymbol{u}\in U}H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p}).$$

*If the final state, $\boldsymbol{x}(t_f)$, is free, then in addition to the above conditions it is required that the following end-point condition is satisfied,*

$$\boldsymbol{p}(t_f)=\nabla_{\boldsymbol{x}}\Phi|_{t=t_f}.$$

The equation,

$$\dot{\boldsymbol{p}} = -\left(\frac{\partial H}{\partial \boldsymbol{x}}\right)^{\top} \tag{8}$$

is called in the literature the *adjoint* or *costate* equation.

We illustrate the above theorem with the following well-known example.

**Example 1** We consider a controlled object modeled by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \end{aligned}$$

with the performance index

$$J = \int_0^{t_f} dt.$$

The control is required to satisfy

$$|u(t)| \leq 1$$

for all $t \in [0, t_f]$. This constraint means that the control must have magnitude no greater than 1. Our objective is to find admissible control that minimizes $J$ and transfers the system from a given initial state $\boldsymbol{x}_0$ to the origin.

We begin by finding the Hamiltonian function for the problem,

$$\begin{aligned} H &= 1 + \boldsymbol{p}^{\top}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u) \\ &= 1 + \begin{bmatrix} p_1 & p_2 \end{bmatrix} \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right) \\ &= 1 + p_1 x_2 + p_2 u. \end{aligned}$$

The costate equations are

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = -\begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -p_1 \end{bmatrix}.$$

Solving the costate equations yields

$$p_1 = d_1 \quad \text{and} \quad p_2 = -d_1 t + d_2,$$

where $d_1$ and $d_2$ are integration constants. We now find an admissible control minimizing the Hamiltonian,

$$\begin{aligned} \arg_u \min H &= \arg_u \min(1 + p_1 x_2 + p_2 u) \\ &= \arg_u \min(p_2 u) \\ &= \begin{cases} u(t) = 1 & \text{if} \quad p_2 < 0 \\ u(t) = 0 & \text{if} \quad p_2 = 0 \\ u(t) = -1 & \text{if} \quad p_2 > 0. \end{cases} \end{aligned}$$

Hence,

$$u^*(t) = -\operatorname{sign}(p_2^*) = -\operatorname{sign}(-d_1 t + d_2),$$

where the signum function is defined as

$$\operatorname{sign}(z) = \begin{cases} 1 & \text{if} \quad z > 0 \\ 0 & \text{if} \quad z = 0 \\ -1 & \text{if} \quad z < 0. \end{cases}$$

Thus, the optimal control law is piecewise constant taking the values 1 or $-1$. This control law has at most two intervals of constancy because the argument of the sign function is a linear function, $-d_1 t + d_2$, that changes its sign at most once. This type of control is called a *bang-bang control* because it switches back and forth between its extreme values. System trajectories for $u = 1$ and $u = -1$ are families of parabolas. Segments of the two parabolas through the origin form the switching curve,

$$x_1 = -\frac{1}{2} x_2^2 \operatorname{sign}(x_2).$$

This means that if an initial state is above the switching curve, then $u = -1$ is used until the switching curve is reached. Then, $u = 1$ is used to reach the origin. For an initial state below the switching curve, the control $u = 1$ is used first to reach the switching curve, and then the control is switched to $u = -1$. We implement the above control action as

$$u = -\operatorname{sign}(v) = \begin{cases} 1 & \text{if} \quad v < 0 \\ 0 & \text{if} \quad v = 0 \\ -1 & \text{if} \quad v > 0, \end{cases}$$

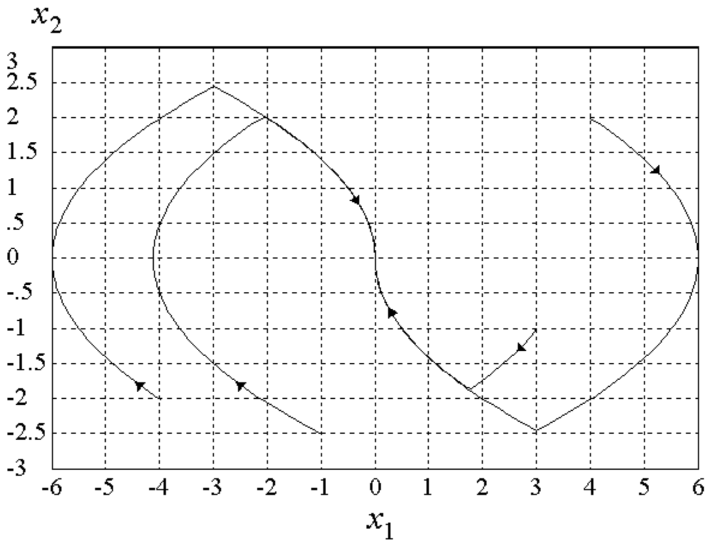


Figure 1: A phase-plane portrait of the time optimal closed-loop system of Example 1.

where $v = v(x_1, x_2) = x_1 + \frac{1}{2}x_2^2\mathrm{sign}(x_2) = 0$ is the equation describing the switching curve. We can use the above equation to synthesize a closed-loop system such that starting at an arbitrary initial state in the state plane, the trajectory will always be moving in an optimal fashion towards the origin. Once the origin is reached, the trajectory will stay there. A phase portrait of the closed-loop time optimal system is given in Figure 1.

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# References


[1] D. G. Luenberger. *Introduction to Dynamic Systems: Theory, Models, and Applications.* John Wiley & Sons, New York, 1979.