Solutions

THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 3413
ANALYSIS AND DESIGN OF CONTROL SYSTEMS

ASSESSMENT EXAM
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The objective of this exercise is to assess your knowledge of the course prerequisites. This will give me a good indication of your background. You are not supposed to do so much work for this exam—it's only assessment, remember! I'll read your solutions and try to fill the corresponding gaps in class. If you do not know the answer to any of the questions, leave it blank.

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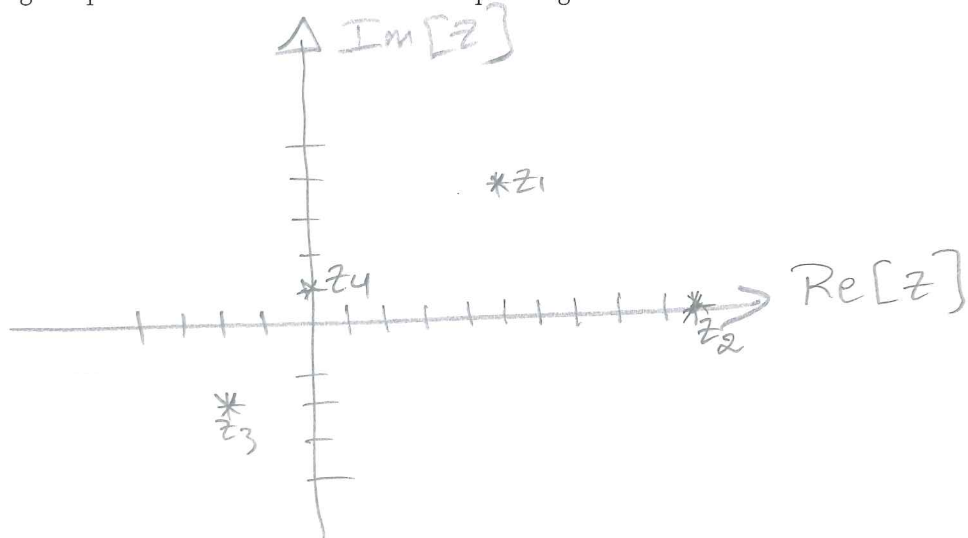
1. Plot and label the following complex numbers and find their corresponding absolute values.
   (a) $z_1 = 5 + 4j$
   (b) $z_2 = 10$
   (c) $z_3 = -2 - 2j$
   (d) $z_4 = j$



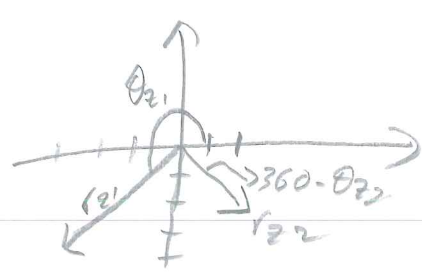
2. Convert the following complex numbers from rectangular to **polar and exponential** forms.
   (a) $z_1 = -3 - 4j$
   (b) $z_2 = 2 - 2j$

$\underbrace{z = x + yj}_{\text{cartesian /rectangular}} \longrightarrow r, \varphi$

$r = \sqrt{x^2 + y^2}$

$\varphi = \arctan'(y, x)$

$\underbrace{z = x + yj}_{\text{cartesian /rectangular}} \longrightarrow z = re^{j\theta} = r(\cos\theta + j\sin\theta)$



3. Simplify the following expressions for $z_1 = 4 + 2j$, $z_2 = 6 - 5j$, and $z_3 = -1 + 4j$.

   (a) $z_1\overline{(z_2 - z_3)}$

   (b) $\dfrac{z_2^*}{z_1 z_3}$

(a) $(4+2j)\overline{(6-5j+1-4j)} = (4+2j)\overline{(7-9j)}$

$= (4+2j)(7+9j)$

$= 28 + (36+14)j - 18$

$\Longrightarrow z_1\overline{(z_2 - z_3)} = 10 + 50j$

(b) $\dfrac{z_2^*}{z_1 z_3} = \dfrac{6+5j}{(4+2j)(-1+4j)} = \dfrac{6+5j}{-12+14j} \cdot \dfrac{-12-14j}{-12-14j}$

$= \dfrac{(6+5j)(-12-14j)}{(12^2+14^2)}$

$= \dfrac{1}{340}(-2-144j)$

4. The Laplace transform, $F(s)$, of a time-function $f(t)$ is given by:

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

Applying the definition, find the Laplace transform of the following functions:

(a) $f_1(t) = 10e^{3t}$

(b) $f_2(t) = 5 + 2e^{-2t}$

(a) $\mathcal{L}[10e^{3t}] = F_1(s) = \int_0^{\infty} 10e^{3t} \cdot e^{-st}\, dt$

$$= 10 \int_0^{\infty} e^{3t} e^{-st}\, dt$$

$$= 10 \int_0^{\infty} e^{-(s-3)t}\, dt$$

$$= \frac{-10}{s-3} \cdot -1 \int_0^{\infty} (s-3)\, e^{-(s-3)t}\, dt$$

$$= \frac{-10}{s-3} \left. e^{-(s-3)} \right]_0^{\infty}$$

$$= \frac{-10}{s-3} \left[e^{-(s-3)t}\Big|_{t=\infty} - e^{-(s-3)t}\Big|_{t=0} \right]$$

$$= \frac{-10}{s-3} [0 - 1] = \frac{10}{s-3}$$

(b) Similar to (a), $\mathcal{L}[f_2(t)] = \frac{5}{s} + \frac{2}{s+2}$

5. Using the provided table, find the Laplace transform of

$$f_3(t) = 2t\cos(5t) + 3\sinh(2t) - 4e^{2t}.$$

Do note compute it via the definition of Laplace transform.

$$\mathcal{L}\left[2t\cos(5t)\right] = 2\mathcal{L}\left[t\cos(\underset{\alpha=5}{5}t)\right] = 2\left[\frac{s^2-\alpha^2}{(s^2+\alpha^2)^2}\right]$$

$$= 2\left(\frac{s^2-25}{(s^2+25)^2}\right);$$

$$\mathcal{L}\left[3\sinh(2t)\right] = \frac{3\cdot 2}{s^2-4} = \frac{6}{s^2-4};$$

$$\mathcal{L}\left[-4e^{2t}\right] = \frac{-4}{s-2}; \Rightarrow \mathcal{L}\left[f_3(t)\right] = \square + \square + \square$$

6. Solve this indefinite integral:

$$\int \frac{1}{1+e^x}dx.$$

Hint: 1 apple + 1 orange - 1 orange = 1 apple ;)

$$\int \frac{1}{1+e^x}dx \underset{\text{(using the hint)}}{=} \int \frac{\overbrace{1}^{\text{apple}} + \overbrace{e^x}^{\text{orange}} - \overbrace{e^x}^{\text{orange ;)}}}{1+e^x}dx$$

$$= \int \frac{1+e^x}{1+e^x}dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \left(\int \frac{e^x}{1+e^x}dx\right) \rightarrow \int \frac{u'}{u} = \ln|u|$$

$$\Rightarrow \int \frac{1}{1+e^x}dx = x - \ln|1+e^x| + C$$

7. Using the provided table, find the inverse Laplace transform of

$$F_4(s) = \frac{2}{s} - \frac{3}{2s-5}.$$

$F_4(s) = 2\left(\frac{1}{s}\right) - \frac{3}{2}\left(\frac{1}{s - 5/2}\right)$

$\rightarrow \mathcal{L}^{-1}[F_4(s)] = 2u(t) - \frac{3}{2} e^{5/2 t}$

8. Using Laplace transforms, find the transfer function of the system governed by this differential equation:

$$y''(t) + 4y'(t) + 3y(t) = 2u(t),$$

and $y(0) = 1, y'(0) = 0$. You should use this property:

$$\boxed{\mathcal{L}(y'(t)) = sY(s) - y(0), \quad \mathcal{L}(y''(t)) = s^2Y(s) - sy(0) - sy'(0).}$$

Note that the transfer function of a system with input $u(t)$ and output $y(t)$ is given by:

$$H(s) = \frac{Y(s)}{U(s)}.$$

$\mathcal{L}\left[y''(t) + 4y'(t) + 3y(t) = 2u(t) \right]$

$\Rightarrow \mathcal{L}[y''(t)] + 4\mathcal{L}[y'(t)] + 3\mathcal{L}[y(t)] = 2\mathcal{L}[u(t)]$

$\Rightarrow s^2Y(s) - s + 4sY(s) + 3Y(s) = 2U(s)$

$\Rightarrow (s^2 + 3s + 3)Y(s) = (2)U(s)$

$\Rightarrow \frac{Y(s)}{U(s)} = \frac{2}{s^2 + 3s + 3}$

Table of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{\left(s^2+a^2\right)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{\left(s^2+a^2\right)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s\left(s^2+3a^2\right)}{\left(s^2+a^2\right)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n e^{at}, \quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	e^{-cs}
27.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	28.	$u_c(t)g(t)$	$e^{-cs}\mathcal{L}\{g(t+c)\}$
29.	$e^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^{\infty} F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			