THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 3413
ANALYSIS AND DESIGN OF CONTROL SYSTEMS

HOMEWORK # 2 SOLUTIONS
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The objective of this homework is to test your understanding of the content of Module 2—Laplace transforms, transfer functions, partial fraction expansion, and ODE solutions. Provide neat solutions, with well-written answers. You need to have the Laplace transform table around you when you're doing the homework. Due date of the homework is: **Thursday, January 28th, 2016.**

You have to upload a scanned version of your solutions on Blackboard. If you don't have a scanner around you, you can use Cam Scanner—a mobile app that scans images in a neat way, as if they're scanned through a copier. Here's the link for Cam Scanner: https://www.camscanner.com/user/download.

1. Using Laplace transforms, solve the following differential equation for $y(t)$:

$$y'(t) - y(t) = e^{3t},$$

given that the initial value for $y(t)$ is $y(0) = 2$.

Solution:

Applying the LT, we get:

$$sY(s) - y(0) - Y(s) = \frac{1}{s-3}.$$

Hence,

$$Y(s) = \frac{2}{s-1} + \frac{1}{(s-1)(s-3)} = \frac{2}{s-1} + \frac{A}{s-1} + \frac{B}{s-3}.$$

Via partial fraction expansion, $A = -0.5, B = 0.5$. Therefore, we can obtain $y(t)$ by taking the inverse Laplace transform:

$$y(t) = 2e^t - 0.5e^t + 0.5e^{3t} = 1.5e^t + 0.5e^{3t}.$$

2. Using Laplace transforms, solve the following differential equation for $y(t)$:

$$y''(t) - 10y'(t) + 9y(t) = 5t,$$

given that $y(0) = -1$ and $y'(0) = 2$. Verify your answers on MATLAB via the `ilaplace` command.

Solution:

We can immediately apply the Laplace transform the above ODE:

$$(s^2 - 10s + 9)Y(s) + s - 2 - 10 = \frac{5}{s^2} \Rightarrow Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}.$$

Using the techniques we discussed in class, we can find the residuals of $Y(s)$. First, we start with s^2 and $s-1, s-9$. These are the easiest terms to find:

$$B = s^2Y(s)|_{s=0} = \frac{5}{9} \approx 0.55, C = (s-9)Y(s)|_{s=9} = \frac{31}{81} \approx 0.38, C = (s-1)Y(s)|_{s=1} = -2.$$

Using these coefficients, we can find A by substituting $s = 2$ (or any other number that isn't a pole). Consequently, $A = \frac{50}{81} \approx 0.61$. Therefore:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}\right] = 0.61 + 0.55t + 0.38e^{9t} - 2e^t.$$

To solve this problem with MATLAB, we can write this short script:

```
>> syms s t
>> Y=(5+12*s^2-s^3)/((s^2)*(s-9)*(s-1))

Y =

(- s^3 + 12*s^2 + 5)/(s^2*(s - 1)*(s - 9))

>> y=ilaplace(Y)

y =

(5*t)/9 + (31*exp(9*t))/81 - 2*exp(t) + 50/81

% This verifies our answer.
```

3. For this differential equation:

$$y''(t) - 6y'(t) + 15y(t) = 2u(t),$$

solve the following problems:

(a) The transfer function $\frac{Y(s)}{U(s)}$.

(b) The poles and zeros (if any) of the transfer function.

(c) Given that $u(t) = \sin(3t)$, $y(0) = -1$, $y'(0) = -4$, find $y(t)$ using partial fraction expansion. You might need to solve multiple linear equations with multiple unknowns. Do not panic.

(d) Verify your answers on MATLAB via the `ilaplace` command.

Solution:

(a) For zero initial conditions, the transfer function is:

$$\frac{Y(s)}{U(s)} = \frac{2}{s^2 - 6s + 15}.$$

(b) Poles: $p_{1,2} = 3 \pm 2.45j$, Zeros: none.

(c) Given that the input $u(t)$ is equal to $\sin(3t)$, we can find the Laplace transform of $u(t)$:

$$U(s) = \frac{3}{s^2 + 9}.$$

After considering the initial conditions and the control input, we obtain:

$$(s^2 - 6s + 15)Y(s) + s - 2 = \frac{2 \cdot 3}{s^2 + 9} \Rightarrow Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2 + 9)(s^2 - 6s + 15)} = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{s^2 - 6s + 15}.$$

Using partial fraction expansion, we can find A, B, C, and D. You can think of plenty of ways to solve this problem. The most trivial way is to substitute values for s and obtain a system of four equations and for unknowns. You can also find the common denominator of the LHS of the above equations and come up with equivalent equations. Regardless, your solution should be unique:

$$A = \frac{1}{10}, B = \frac{1}{10}, C = -\frac{11}{10}, D = \frac{5}{2}.$$

Thus,

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s + 25}{s^2 - 6s + 15}\right).$$

Using the results from the Laplace transform table and the example in class, we obtain:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{10}\left(\cos(3t) + \frac{1}{3}\sin(3t) - 11e^{3t}\cos(\sqrt{6}t) - \frac{8}{\sqrt{6}}e^{3t}\sin(\sqrt{6}t)\right).$$

(d) To solve this problem via MATLAB, we can write the following script:

```
>> syms s t
>> Y=(-s^3+2*s^2-9*s+24)/((s^2+9)*(s^2-6*s+15))

Y =

-(s^3 - 2*s^2 + 9*s - 24)/((s^2 + 9)*(s^2 - 6*s + 15))

>> y = ilaplace(Y)

y =

cos(3*t)/10 + sin(3*t)/30 - (11*exp(3*t)*(cos(6^(1/2)*t)
+ (4*6^(1/2)*sin(6^(1/2)*t))/33))/10

% This verifies our findings.
```