THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5243
INTRODUCTION TO CYBER-PHYSICAL SYSTEMS

HOMEWORK # 3
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Homework Instructions:

1. Type your solutions in the LaTeX homework template file. Otherwise, you can use any other typesetting tool or you can provide handwritten solutions, assuming everything is clear.
2. **Due date:** Friday, September 11th, @ 5:00pm on Blackboard, **AND** drop off a copy of your solutions (slip it under the office door if I'm away).
3. **Collaboration policy:** you can collaborate with your classmates, under the assumption that everyone is required to write their own solutions. If you choose to collaborate with anyone, list their name(s).
4. You don't show your work $\Rightarrow$ You don't get credit.
5. Solutions that are unclear won't be graded.
6. Before you start with this homework assignment, make sure that you have grasped the content of Module 03. Most of the answers for this problem set are simple applications of the theories and results in Module 03.

Problem 1 — Solution of a DTLTI System

Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

1. Given that $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the control is equal to zero for all k, determine $x(0)$.
2. Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k}1^+(k)$ and $x(0) = 0$.

Problem 2 — Solution of a DTLTI System (2)

Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}}_{D} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

1. Find a general expression for D^k.
2. Find A^k.
3. Compute $x(k)$ if the control input is null.
4. Computer $x(k)$ if the initial conditions are null and the control input is $u(k) = 2^k 1^+(k)$ and $\lambda_1 = 4$.

Problem 3 — Solution of a CTLTI System

Given a CTLTI model,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = T \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} T^{-1}, \boldsymbol{B} = T \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix}, a \neq 0, b \neq 0.$$

1. Determine e^{At}.
2. Find $e^{A(t-\tau)}B$.
3. Given that $u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{bt} 1^{+}(t)$ and $x(2) = T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $x(0)$.

Problem 4 — State-Feedback Controller Design

Given a CTLTI model,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Assume that a linear state-feedback controller of this form

$$u(t) = Kx(t) = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix} x(t)$$

is used as a control input.

1. Find $A + BK$ in terms of $k_1, \ldots, k_8$.
2. Find K such that $A + BK$ is block-diagonal (i.e., formed by two blocks of 2-by-2 matrices on the diagonal and zeros elsewhere.) and the first block has eigenvalues (2,3) and the second block has eigenvalues (0,1).

Problem 5 — Linear Systems Properties

Consider the discrete-time LTI dynamical system:

$$x(k+1) = Ax(k) + Bu(k), \;\; y(k) = Cx(k),$$

where

$$A^k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}.$$

1. Is the system controllable?
2. What is the set of reachable space in 3 time-steps, assuming that the initial condition is $x(0) = 0$? In other words, what is a set that contains all possible values of $x(3)$ given some control function $u(k)$ for $k = 0, 1, 2$?
3. Is the system observable?

4. Find the unobservable subspace, if any.
5. Is the system asymptotically stable?
6. The system is stabilizable. True or False?
7. The system is detectable. True or False?
8. The transfer function of a DTLTI system is given by: $H(z) = C(zI - A)^{-1}B$. Compute the transfer function.

Problem 6 — Stability of Nonlinear Systems

Consider the following nonlinear system:

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t)(x_1^2(t) - 1) \\
\dot{x}_2(t) &= x_2^2(t) + x_1(t) - 3
\end{aligned}
$$

1. Find all the equilibrium points of the nonlinear system.
2. Determine the stability of the system around each equilibrium point, if possible. You can verify your solutions by plotting phase portraits on MATLAB.