HOMEWORK # 3 SOLUTIONS
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Problem 1 — Solution of a DTLTI System

Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

1. Given that $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the control is equal to zero for all k, determine $x(0)$.
2. Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k}1^+(k)$ and $x(0) = 0$.

Solutions:

1. Since $u(k) = 0$, then:

$$x(k+1) = Ax(k) \Rightarrow x(2) = A^2 x(0) \Rightarrow x(2) = \begin{bmatrix} 2a^{2-1} & 1 \\ 0 & a^2 \end{bmatrix} x(0)$$

$$\Rightarrow x(0) = \begin{bmatrix} 2a & 1 \\ 0 & a^2 \end{bmatrix}^{-1} x(2) = \frac{1}{2a^3} \begin{bmatrix} a^2 - 1 \\ 2a \end{bmatrix} = \begin{bmatrix} \frac{1}{2a} - \frac{1}{2a^3} \\ \frac{1}{a^2} \end{bmatrix}$$

2. From Module 03 notes,

$$x(n) = \sum_{k=0}^{n-1} A^{n-1-k} B u(k) = \sum_{k=0}^{n-1} A^k B u(n-1-k) = \sum_{k=0}^{n-1} A^k B a^{k-n+1} = \sum_{k=0}^{n-1} \begin{bmatrix} ka^{k-1} a^{k-n+1} \\ 0 \end{bmatrix}.$$

Hence,

$$x(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = \begin{bmatrix} a^{-n} \sum_{k=0}^{n-1} k(a^2)^k \\ 0 \end{bmatrix} = \begin{bmatrix} a^{-n} \frac{d}{da} \left(\frac{1-(a^2)^n}{1-a^2} \right) \\ 0 \end{bmatrix}.$$

Problem 2 — Solution of a DTLTI System (2)

Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}}_{D} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

1. Find a general expression for D^k.
2. Find A^k.
3. Compute $x(k)$ if the control input is null.

4. Computer $x(k)$ if the initial conditions are null and the control input is $u(k) = 2^k 1^+(k)$ and $\lambda_1 = 4$.

Solutions:

1. $D^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix}$

2. $A^k = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$

3. $x_{\text{zisr}}(k) = A^k x(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k\lambda_1^{k-1} \\ \lambda_1^k \end{bmatrix}$

4. The zero-state state response can be written as:

$$x_{\text{zssr}}(n) = \sum_{k=0}^{n-1} A^{n-1-k} B u(k) = \sum_{k=0}^{n-1} A^k B u(n-1-k)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sum_{k=0}^{n-1} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} u(n-1-k)$$

$$= 2^n \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sum_{k=0}^{n-1} \begin{bmatrix} 2^k \\ 0 \end{bmatrix} = (2^{2n} - 2^n) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem 3 — Solution of a CTLTI System

Given a CTLTI model,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = T \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} T^{-1}, \boldsymbol{B} = T \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix}, a \neq 0, b \neq 0.$$

1. Determine e^{At}.

2. Find $e^{A(t-\tau)}B$.

3. Given that $u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{bt} 1^+(t)$ and $x(2) = T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $x(0)$.

Solutions:

1. $e^{At} = T \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{bt} \end{bmatrix} T^{-1}$

2. $e^{A(t-\tau)}B = T \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & be^{b(t-\tau)} \end{bmatrix}$

3. You can always go backward in an integration:

$$x(0) = e^{A(0-2)t} + \int_2^0 e^{A(t-\tau)} Bu(\tau) d\tau = T \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-2a} & 0 \\ 0 & 0 & e^{-2b} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + T \begin{bmatrix} 0 \\ 0 \\ be^{b0} \end{bmatrix} (0-2)$$

$$= T \left(\begin{bmatrix} 1 \\ 2e^{-2a} \\ 3e^{-2b} \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \right)$$

Problem 4 — State-Feedback Controller Design

Given a CTLTI model,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Assume that a linear state-feedback controller of this form

$$u(t) = Kx(t) = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix} x(t)$$

is used as a control input.

1. Find $A + BK$ in terms of $k_1, \ldots, k_8$.
2. Find K such that $A + BK$ is block-diagonal (i.e., formed by two blocks of 2-by-2 matrices on the diagonal and zeros elsewhere.) and the first block has eigenvalues (2,3) and the second block has eigenvalues (0,1).

Solutions:

1. $A + BK = \begin{bmatrix} k_1 & 1+k_2 & 1+k_3 & 1+k_4 \\ 1 & 0 & 0 & 0 \\ 1+k_5 & 1+k_6 & k_7 & k_8 \\ 1+k_5 & 1+k_6 & 1+k_7 & k_8 \end{bmatrix}$

2. Since we want $A + BK$ to be block-diagonal, then we need $k_3 + 1 = k_4 + 1 = \ldots = 0$, or $k_3 = k_4 = k_5 = k_6 = -1$. Also, given that the first block has (2,3) as assigned eigenvalues, then using the pole-assignment procedure from Module 03, we can find k_1 and k_2. The characteristic polynomial of the first block is

$$\pi^{(1)}_{(A+BK)} = \lambda^2 - k_1\lambda - (1 + k_2) = 0 \Rightarrow k_1 = 5, k_2 = -7.$$

Similarly, $k_8 = 0$ and $k_7 = 1$. Thus:

$$F = \begin{bmatrix} 5 & -7 & -1 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}.$$

Problem 5 — Linear Systems Properties

Consider the discrete-time LTI dynamical system:

$$x(k+1) = Ax(k) + Bu(k), \;\; y(k) = Cx(k),$$

where

$$A^k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}.$$

1. Is the system controllable?
2. What is the set of reachable space in 3 time-steps, assuming that the initial condition is $x(0) = 0$? In other words, what is a set that contains all possible values of $x(3)$ given some control function $u(k)$ for $k = 0, 1, 2$?
3. Is the system observable?

4. Find the unobservable subspace, if any.

5. Is the system asymptotically stable?

6. The system is stabilizable. True or False?

7. The system is detectable. True or False?

8. The transfer function of a DTLTI system is given by: $H(z) = C(zI - A)^{-1}B$. Compute the transfer function.

Solutions:

1. Controllability matrix of the given system is:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{C}) = 3 \Rightarrow \textbf{system is not controllable}$$

2. Set of states that can be reached from a zero initial state condition is given by the subspace of $\mathbb{R}^4$ spanned by only the first three columns of the controllability matrix $\mathcal{C}$.

3. Observability matrix of the given system is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{O}) = 2 \Rightarrow \textbf{system is not observable}$$

4. As introduced in class, the set of unobservable subspace is the *null-space* of $\mathcal{O}$, that is:

$$\text{null}(\mathcal{O}) := \left\{ x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \in \mathbb{R}^4 \mid x_2 = x_3 = 0 \right\}$$

5. System is not asymptotically stable; A has two eigenvalues at $-1, 1$, and for discrete systems the eigenvalues at the borders of the unit disk makes the system marginally stable (given that the size of Jordan block is not greater than 1), not asymptotically stable.

6. False — the unstable eigenvalue -1 is uncontrollable (PBH test).

7. False — the unstable eigenvalue -1 is unobservable (PBH test).

8. $H(z) = C(zI - A)^{-1}B = \dfrac{1}{z}$.

Problem 6 — Stability of Nonlinear Systems

Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t)(x_1^2(t) - 1) \\ \dot{x}_2(t) &= x_2^2(t) + x_1(t) - 3 \end{aligned}$$

1. Find all the equilibrium points of the nonlinear system.

2. Determine the stability of the system around each equilibrium point, if possible. You can verify your solutions by plotting phase portraits on MATLAB.

Solutions:

1. Setting the state-dynamics to zero, we can find the equilibrium points. There are 5 equilibrium points for the given system, listed as follows:

$$x_e = \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 & 3 \\ \sqrt{2} & -\sqrt{2} & 2 & -2 & 0 \end{bmatrix}.$$

2. The stability of the system around an equilibrium point is determined by evaluating the Jacobian matrix $Df(x)$ around each equilibrium point and finding its eigenvalues:

$$Df(x) = \begin{bmatrix} 2x_1x_2 & x_1^2 - 1 \\ 1 & 2x_2 \end{bmatrix}.$$

The only equilibrium point that yields a stable $Df(x_e)$ matrix is $x_e^{(2)} = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$, giving $\lambda_1 = \lambda_2 = -2\sqrt{2}$ as the two stable eigenvalues.