HOMEWORK # 3
Hafez Bazrafshan
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To facilitate checking, questions are color-coded blue and pertinent answers follow in regular font. The printed version however, will be in black and white.

Problem 1 — Solution of a DTLTI System

Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

1. Given that $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the control is equal to zero for all k, determine $x(0)$.
2. Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k}1^{+}(k)$ and $x(0) = 0$.

Response.

1. Using the dynamical system model and the fact that $u(k) = 0$ for all k, we see that the following holds:

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) = Ax(1) + \mathbf{0} = Ax(1) \\ x(1) &= Ax(0) + Bu(0) = Ax(0) + \mathbf{0} = Ax(0) \\ \Rightarrow x(2) &= Ax(1) = A(Ax(0)) = A^2x(0). \end{aligned} \quad \text{(P1-1)}$$

According to (P1-1), we now need to calculate A^2:

$$A^2 = \begin{bmatrix} 2a^{2-1} & 1 \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} 2a & 1 \\ 0 & a^2 \end{bmatrix}. \quad \text{(P1-2)}$$

Determining $x(0)$ now boils down to finding the inverse of A^2 from (P1-2) and multiplying it by $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$$x(0) = (A^2)^{-1}x(2) = \begin{bmatrix} \frac{1}{2a} & -\frac{1}{2a^3} \\ 0 & \frac{1}{a^2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2a^3} \begin{bmatrix} a^2 - 1 \\ 2a \end{bmatrix}. \quad \text{(P1-3)}$$

2. Since the initial condition, i.e., $x(0) = 0$ is given, we can find the general expression of $x(n)$ by:

$$x(n) = \sum_{i=0}^{n} A^{n-1-i}Bu(i) = \sum_{i=0}^{n} A^{n-1}A^{-i} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a^{-i} = A^{n-1} \sum_{i=0}^{n} A^{-i} \begin{bmatrix} 1 \\ 0 \end{bmatrix} a^{-i} \quad \text{(P1-4)}$$

Equation (P1-4) is a general expression for $x(n)$. *From here on out in this problem, I will assume that A^k for all k is given by the problem statement (notice that this is not true for $k < 0$, but since the problem hasn't stated this, it gets a little confusing)*. Notice that $A^{-i} = \begin{bmatrix} (-i)a^{-i-1} & 0 \\ 0 & a^{-i} \end{bmatrix}$ we will obtain:

$$x(n) = A^{n-1} \begin{bmatrix} \sum_{i=0}^{n} (-i)a^{-2i-1} \\ 0 \end{bmatrix} \quad \text{(P1-5)}$$

Let's assume $G(n) = \sum_{i=0}^{n}(-i)a^{-2i-1}$. Next, we will show how to calculate $G(n)$. Consider a function $f(x) = \sum_{i=0}^{n}(x^{-2})^i$ over $x \in \mathbb{R}/\{0,1\}$. This function is differentiable over its domain. Interestingly, $f'(x) = \frac{df(x)}{x} = \sum_{i=0}^{n} -2ix^{-2i-1}$. Hence we can establish that

$$f'(a) = \sum_{i=0}^{n} -2ia^{-2i-1} = 2G(n). \tag{P1-6}$$

Notice that $f(x) = \sum_{i=0}^{n}(x^{-2})^i$ is actually a geometric finite sum and has a closed form answer in the domain defined above. Concisely, we have that:

$$f(x) = \sum_{i=0}^{n}(x^{-2})^i = \frac{1-(x^{-2})^{n+1}}{1-(x^{-2})} \tag{P1-7}$$

From (P1-7) one can calculate $f'(x)$:

$$f'(x) = \frac{(2n+2)x^{-2n+1} - 2nx^{-2n-1}}{(x^2-1)^2}. \tag{P1-8}$$

From (P1-6):

$$G(n) = \frac{f'(a)}{2} = \frac{(2n+2)a^{-2n+1} - 2nx^{-2n-1}}{2(a^2-1)^2}. \tag{P1-9}$$

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Using (P1-9), (P1-5), and by replacing $A^{n-1} = \begin{bmatrix}(n-1)a^{n-2} & 1 \\ 0 & a^{n-1}\end{bmatrix}$ we will obtain the final form for $x(n)$:

$$x(n) = A^{n-1}\begin{bmatrix}\sum_{i=0}^{n}(-i)a^{-2i-1} \\ 0\end{bmatrix} = \begin{bmatrix}(n-1)a^{n-2} & 1 \\ 0 & a^{n-1}\end{bmatrix}\begin{bmatrix}G(n) \\ 0\end{bmatrix} = \begin{bmatrix}\frac{(n-1)(2n+2)a^{-n-1} - 2n(n-1)a^{-n-3}}{2(a^2-1)^2} \\ 0\end{bmatrix}. \tag{P1-10}$$

Problem 2 — Solution of a DTLTI System (2)

Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 1 \\ 0 & \lambda_1\end{bmatrix}}_{D}\begin{bmatrix}0.5 & 0.5 \\ 0.5 & -0.5\end{bmatrix}, B = \begin{bmatrix}2 \\ 2\end{bmatrix}, x(0) = \begin{bmatrix}2 \\ -2\end{bmatrix}.$$

1. Find a general expression for D^k.
2. Find A^k.
3. Compute $x(k)$ if the control input is null.
4. Computer $x(k)$ if the initial conditions are null and the control input is $u(k) = 2^k 1^+(k)$ and $\lambda_1 = 4$.

Response.

1. For $k = 0$, $D^k = I_2$ where I_2 denotes the identitiy matrix. Moreover, $k = 1$ is also trivially:

$$D = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}. \tag{P2-1}$$

By induction, we will prove that for $k \geq 1$ the following is true:

$$D^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix}. \tag{P2-2}$$

$$(K = 1): \quad D^1 = \begin{bmatrix} \lambda_1^1 & 1 \times \lambda_1^{1-1} \\ 0 & \lambda_1^1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$
$$(K \to K+1): \quad D^{k+1} = D^k \times D = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_1^{k+1} & (k+1)\lambda_1^k \\ 0 & \lambda_1^{k+1} \end{bmatrix} \tag{P2-3}$$

Notice that we cannot use (P2-2) for $k < 0$ directly (since induction assumes $k > 1$). Therefore, for $k < 0$ we will actually set $m = -k > 0$ and do the following:

$$D^k = (D^m)^{-1} = \begin{bmatrix} \lambda_1^m & m\lambda_1^{m-1} \\ 0 & \lambda_1^m \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1^{-m} & -m\lambda_1^{-m-1} \\ 0 & \lambda_1^{-m} \end{bmatrix} = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \tag{P2-4}$$

Therefore,

$$D^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \quad \forall k \in \mathbb{Z}. \tag{P2-5}$$

2. For $k = 0$, we have that $A^0 = I_2$. Let $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Notice that $T^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$. Trivially, for k=1, $A^1 = TDT^{-1}$ and for $k >= 2$:

$$A^k = \underbrace{A \times \ldots \times A}_{\text{k times}} = TDT^{-1} \times \ldots \times TDT^{-1} = TD^kT^{-1} \tag{P2-6}$$

since $TT^{-1} = I_2$. For $k < 0$, let $m = -k > 0$ and we will have:

$$A^k = A^{-m} = (A^m)^{-1} = (TD^mT^{-1})^{-1} = T(D^m)^{-1}T^{-1} = TD^{-m}T^{-1} = TD^kT^{-1}. \tag{P2-7}$$

In conclusion, we have that:

$$A^k = TD^kT^{-1} \quad \forall k \in \mathbb{Z} \tag{P2-8}$$

where D^k is given by (P2-5).

3. When the input is null:

$$x(k) = A^k x(0) = TD^kT^{-1}x(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$
$$x(k) = \begin{bmatrix} 2\lambda_1^k + (1+k)\lambda_1^{k-1} \\ -2\lambda_1^k + (1+k)\lambda_1^{k-1} \end{bmatrix}. \tag{P2-9}$$

4. We use convolution.

$$\begin{aligned} x(k) &= \sum_{i=0}^{k} A^{k-1-i}Bu(i) = \sum_{i=0}^{k} A^{k-1-i} \begin{bmatrix} 2 \\ 2 \end{bmatrix} 2^i \\ &= \sum_{i=0}^{k} TD^{k-1-i}T^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} 2^i = T(\sum_{i=0}^{k} D^{k-1-i}2^i)T^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = T(D^{k-1} \sum_{i=0}^{k} D^{-i}2^i)T^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= T(D^{k-1} \sum_{i=0}^{k} (\begin{bmatrix} 4^{-i} & (-i)4^{-i-1} \\ 0 & 4^{-i} \end{bmatrix} 2^i))T^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = TD^{k-1} \begin{bmatrix} \sum_i 2^{-i} & \sum_i (-i)2^{-i-2} \\ \sum_i 0 & \sum_i 2^{-i} \end{bmatrix} T^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{aligned} \tag{P2-10}$$

where we define $\sum_i = \sum\limits_{i=0}^{k}$. By a similar method presented in problem 1 [c.f. (P1-9)] we can calculate:

$$\sum_{i=0}^{k} 2^{-i} = 2(1-(\frac{1}{2})^{k+1}) \tag{P2-11}$$

$$\sum_{i=0}^{k} (-i)2^{-i-2} = 2^k(k-1)+1 \tag{P2-12}$$

Putting everything together:

$$\begin{aligned} x(k) &= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}4^{k-1} & (k-1)4^{k-2}\\ 0 & 4^{k-1}\end{bmatrix}\begin{bmatrix}2(1-(\frac{1}{2})^{k+1}) & 2^k(k-1)+1\\ 0 & 2(1-(\frac{1}{2})^{k+1})\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\ 0.5 & -0.5\end{bmatrix}\begin{bmatrix}2\\ 2\end{bmatrix} \\ &= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}4^{k-1} & (k-1)4^{k-2}\\ 0 & 4^{k-1}\end{bmatrix}\begin{bmatrix}2(1-(\frac{1}{2})^{k+1}) & 2^k(k-1)+1\\ 0 & 2(1-(\frac{1}{2})^{k+1})\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} \\ &= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}4^{k-1} & (k-1)4^{k-2}\\ 0 & 4^{k-1}\end{bmatrix}\begin{bmatrix}2(1-(\frac{1}{2})^{k+1})\\ 0\end{bmatrix} \\ &= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}4^{k-1}2(1-(\frac{1}{2})^{k+1})\\ 0\end{bmatrix} = \begin{bmatrix}4^{k-1}2(1-(\frac{1}{2})^{k+1})\\ 4^{k-1}2(1-(\frac{1}{2})^{k+1})\end{bmatrix}. \end{aligned} \tag{P2-13}$$

Problem 3 — Solution of a CTLTI System

Given a CTLTI model,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = T\begin{bmatrix}0 & 0 & 0\\ 0 & a & 0\\ 0 & 0 & b\end{bmatrix}T^{-1}, \boldsymbol{B} = T\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & b\end{bmatrix}, a \neq 0, b \neq 0.$$

1. Determine e^{At}.
2. Find $e^{A(t-\tau)}B$.
3. Given that $u(t) = \begin{bmatrix}0\\ 1\end{bmatrix} e^{bt}1^{+}(t)$ and $x(2) = T\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$, find $x(0)$.

Response.

1. Since A is *similar* to a diagnoal matrix, we can easily determine e^{At} by invoking the exponential on the diagonal terms, i.e.,

$$e^{At} = T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{at} & 0\\ 0 & 0 & e^{bt}\end{bmatrix}T^{-1}. \tag{P3-1}$$

Notice that (P3-1) holds since $a, b \neq 0$.

2. First, we find $e^{A(t-\tau)}$ by shifting e^{At} and then we multiply.

$$e^{A(t-\tau)} = T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{a(t-\tau)} & 0\\ 0 & 0 & e^{b(t-\tau)}\end{bmatrix}T^{-1}. \tag{P3-2}$$

Therefore,

$$e^{A(t-\tau)}B = T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{a(t-\tau)} & 0\\ 0 & 0 & e^{b(t-\tau)}\end{bmatrix}T^{-1}T\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & b\end{bmatrix} = T\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & be^{b(t-\tau)}\end{bmatrix}. \tag{P3-3}$$

3. We plug in (P3-1) and (P3-3) into the solution of differential equation:

$$
\begin{aligned}
x(2) &= e^{A(2-0)}x(0) + \int_0^2 e^{A(2-\tau)}Bu(\tau)d\tau \\
&= T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{a2} & 0\\ 0 & 0 & e^{b2}\end{bmatrix}T^{-1}x(0) + \int_0^2 T\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & be^{b(2-\tau)}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}e^{b\tau}d\tau \\
&= T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{a2} & 0\\ 0 & 0 & e^{b2}\end{bmatrix}T^{-1}x(0) + \int_0^2 T\begin{bmatrix}0\\ 0\\ be^{b(2-\tau)}\end{bmatrix}e^{b\tau}d\tau \\
&= T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{a2} & 0\\ 0 & 0 & e^{b2}\end{bmatrix}T^{-1}x(0) + \int_0^2 T\begin{bmatrix}0\\ 0\\ be^{2b}\end{bmatrix}d\tau \\
&= T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{a2} & 0\\ 0 & 0 & e^{b2}\end{bmatrix}T^{-1}x(0) + T\begin{bmatrix}0\\ 0\\ 2be^{2b}\end{bmatrix} = T\begin{bmatrix}1\\2\\3\end{bmatrix} \\
\Rightarrow & T\begin{bmatrix}1\\ 2\\ 3-2be^{2b}\end{bmatrix} = T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{a2} & 0\\ 0 & 0 & e^{b2}\end{bmatrix}T^{-1}x(0) \\
\Rightarrow & x(0) = T\begin{bmatrix}1 & 0 & 0\\ 0 & e^{-a2} & 0\\ 0 & 0 & e^{-b2}\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3-2be^{2b}\end{bmatrix} = T\begin{bmatrix}1\\ 2e^{-a2}\\ 3e^{-b2}-2\end{bmatrix}.
\end{aligned}
\tag{P3-4}
$$

Problem 4 — State-Feedback Controller Design

Given a CTLTI model,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = \begin{bmatrix}0 & 1 & 1 & 1\\ 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0\end{bmatrix}, \boldsymbol{B} = \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}.$$

Assume that a linear state-feedback controller of this form

$$u(t) = Kx(t) = \begin{bmatrix}k_1 & k_2 & k_3 & k_4\\ k_5 & k_6 & k_7 & k_8\end{bmatrix}x(t)$$

is used as a control input.

1. Find $A + BK$ in terms of $k_1, \ldots, k_8$.
2. Find K such that $A + BK$ is block-diagonal (i.e., formed by two blocks of 2-by-2 matrices on the diagonal and zeros elsewhere.) and the first block has eigenvalues (2,3) and the second block has eigenvalues (0,1).

Response.

1.

$$A + BK = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 0 \\ k_5 & k_6 & k_7 & k_8 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$$
$$= \begin{bmatrix} k_1 & 1+k_2 & 1+k_3 & 1+k_4 \\ 1 & 0 & 0 & 0 \\ 1+k_5 & 1+k_6 & k_7 & k_8 \\ 1+k_5 & 1+k_6 & 1+k_7 & k_8 \end{bmatrix}. \quad \text{(P4-1)}$$

2. Using (P4-1), in order for $A + BK$ to be composed of 2 block diagonal matrices we must set

$$k_3 = k_4 = k_5 = k_6 = -1. \quad \text{(P4-2)}$$

At this point, we have that:

$$A + BK = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} \quad \text{(P4-3)}$$

where $\boldsymbol{A}_1 = \begin{bmatrix} k_1 & 1+k_2 \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{A}_2 = \begin{bmatrix} k_7 & k_8 \\ 1+k_7 & k_8 \end{bmatrix}$.

In order to have $eig(\boldsymbol{A}_1) = (2, 3)$:

$$det(\begin{bmatrix} k_1 - \lambda & 1+k_2 \\ 1 & 0-\lambda \end{bmatrix}) = (k_1 - \lambda)(-\lambda) - 1 - k_2 = (\lambda - k_1)(\lambda) - 1 - k_2$$
$$= \lambda^2 - k_1\lambda - 1 - k_2 = (\lambda - 2)(\lambda - 3) = \lambda^2 - 5\lambda + 6$$
$$\Rightarrow k_1 = 5, k_2 = -7. \quad \text{(P4-4)}$$

In order to have $eig(\boldsymbol{A}_2) = (2, 3)$:

$$det(\begin{bmatrix} k_7 - \lambda & k_8 \\ 1+k_7 & k_8 - \lambda \end{bmatrix}) = (k_7 - \lambda)(k_8 - \lambda) - k_8 - k_7k_8 = (\lambda - k_7)(\lambda - k_8) - 1 - k_2$$
$$= \lambda^2 - (k_8 + k_7)\lambda - k_8 = (\lambda)(\lambda - 1) = \lambda^2 - \lambda$$
$$\Rightarrow k_7 = 1, k_8 = 0. \quad \text{(P4-5)}$$

Using (P4-2), (P4-4) and (P4-5) the desired K is obtained:

$$K_{\text{desired}} = \begin{bmatrix} 5 & -7 & -1 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}. \quad \text{(P4-6)}$$

Problem 5 — Linear Systems Properties

Consider the discrete-time LTI dynamical system:

$$x(k+1) = Ax(k) + Bu(k), \;\; y(k) = Cx(k),$$

where

$$A^k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}.$$

1. Is the system controllable?
2. What is the set of reachable space in 3 time-steps, assuming that the initial condition is $x(0) = 0$? In other words, what is a set that contains all possible values of $x(3)$ given some control function $u(k)$ for $k = 0, 1, 2$?
3. Is the system observable?
4. Find the unobservable subspace, if any.
5. Is the system asymptotically stable?
6. The system is stabilizable. True or False?
7. The system is detectable. True or False?
8. The transfer function of a DTLTI system is given by: $H(z) = C(zI - A)^{-1}B$. Compute the transfer function.

Response.

1. To figure out whether the system is controllable, the controllability matrix $\mathcal{C}$ can be calculated:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{P5-1}$$

   It is evident that $rank(\mathcal{C}) = 3 < 4$, and therefore the system is not controllable.

2. We can start by calculating $x(3)$ using the previous inputs, i.e.,

$$\begin{aligned} x(0) &= 0 \\ x(1) &= Ax(0) + Bu(0) = Bu(0) \\ x(2) &= Ax(1) + Bu(1) = A[Bu(0)] + Bu(1) = ABu(0) + Bu(1) \\ x(3) &= Ax(2) + Bu(2) = A[ABu(0) + Bu(1)] + Bu(2) = A^2Bu(0) + ABu(1) + Bu(2) \end{aligned} \tag{P5-2}$$

   Assuming generic vectors for $u(0), u(1), u(2)$, (P5-2) clearly shows that the column space of the matrix $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$ will the reachable space of $x(3)$. Therefore, the reachable space will be :

$$C(\begin{bmatrix} B & AB & A^2B \end{bmatrix}) = C(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}) = \begin{bmatrix} u_1 \\ u_2 \\ 0 \\ u_3 \end{bmatrix} \tag{P5-3}$$

   where u_1, u_2, u_3 are independent variables.

3. To assess observability, the observability matrix can be computed:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{P5-4}$$

   It is clear that $rank(\mathcal{O}) = 2$ which means that the system is not observable.

4. The unobservable subspace is the nullspace of O, i.e., we will have to find:

$$\mathcal{N}\{\mathcal{O}\} = \{v | \mathcal{O}v = 0\}. \tag{P5-5}$$

Therefore, we must find all possible $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$:

$$\begin{aligned}
\mathcal{O}v = 0 \Rightarrow & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
& \Rightarrow v_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_2 + v_3 \\ -v_3 \\ v_3 \\ -v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
& \Rightarrow v_2 = v_3 = 0 \Rightarrow v = \begin{bmatrix} v_1 \\ 0 \\ 0 \\ v_4 \end{bmatrix}
\end{aligned} \tag{P5-6}$$

where v_1, v_4 are free variables.

5. We need to find eigenvalues of A.

$$det(A - \lambda I) = \lambda^2(\lambda^2 - 1) = 0 \Rightarrow \lambda = 0, 0, -1, 1. \tag{P5-7}$$

Eigenvalues of A are 0,0,-1,1. Since two eigenvalues are on the unit circle, the system is not asymptotically stable.

6. The system is not stabilizable since the PBH test for $\lambda = -1$ fails:

$$rank(\begin{bmatrix} -I - A & B \end{bmatrix}) = rank(\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 \end{bmatrix}) = 3 < 4. \tag{P5-8}$$

7. The system is not detectable since the PBH test for $\lambda = 1$ fails:

$$rank(\begin{bmatrix} I - A \\ C \end{bmatrix}) = rank(\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}) = 3 < 4. \tag{P5-9}$$

8.

$$\begin{aligned}
ZI - A &= \begin{bmatrix} z & -1 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z+1 & 0 \\ 0 & 0 & -1 & z \end{bmatrix} \\
(ZI - A)^{-1} &= \begin{bmatrix} \frac{1}{z} & \frac{1}{z^2} & 0 & 0 \\ 0 & \frac{1}{z} & 0 & 0 \\ 0 & 0 & \frac{1}{z+1} & 0 \\ 0 & 0 & \frac{1}{z(z+1)} & \frac{1}{z} \end{bmatrix} \\
\Rightarrow C(ZI - A)^{-1}B &= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{z} & \frac{1}{z^2} & 0 & 0 \\ 0 & \frac{1}{z} & 0 & 0 \\ 0 & 0 & \frac{1}{z+1} & 0 \\ 0 & 0 & \frac{1}{z(z+1)} & \frac{1}{z} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{z}.
\end{aligned} \tag{P5-10}$$

Problem 6 — Stability of Nonlinear Systems

Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)(x_1^2(t)-1)\\ \dot{x}_2(t) &= x_2^2(t)+x_1(t)-3\end{aligned}$$

1. Find all the equilibrium points of the nonlinear system.
2. Determine the stability of the system around each equilibrium point, if possible. You can verify your solutions by plotting phase portraits on MATLAB.

1. We find the equilibrium points by:

$$x_2(x_1^2-1)=0 \tag{P5-11}$$

$$x_2^2+x_1-3=0 \tag{P5-12}$$

   Equation (P5-11) yields three cases: $x_2=0$ or $x_1=1$ or $x_1=-1$.
   Substituting these values for equation (P5-12) gives the following equilibrium points :

$$(3,0),(1,+\sqrt{2}),(1,-\sqrt{2}),(-1,2),(-1,-2).$$

2. The jacobian is

$$Df(x)=\begin{bmatrix}2x_2x_1 & x_1^2-1\\ 1 & 2x_2\end{bmatrix}. \tag{P5-13}$$

   Plugging in the values for equilibrium points yields:

$$Df(3,0)=\begin{bmatrix}0 & 8\\ 1 & 0\end{bmatrix}\text{ not negative definite} \Rightarrow \textit{unstable}$$

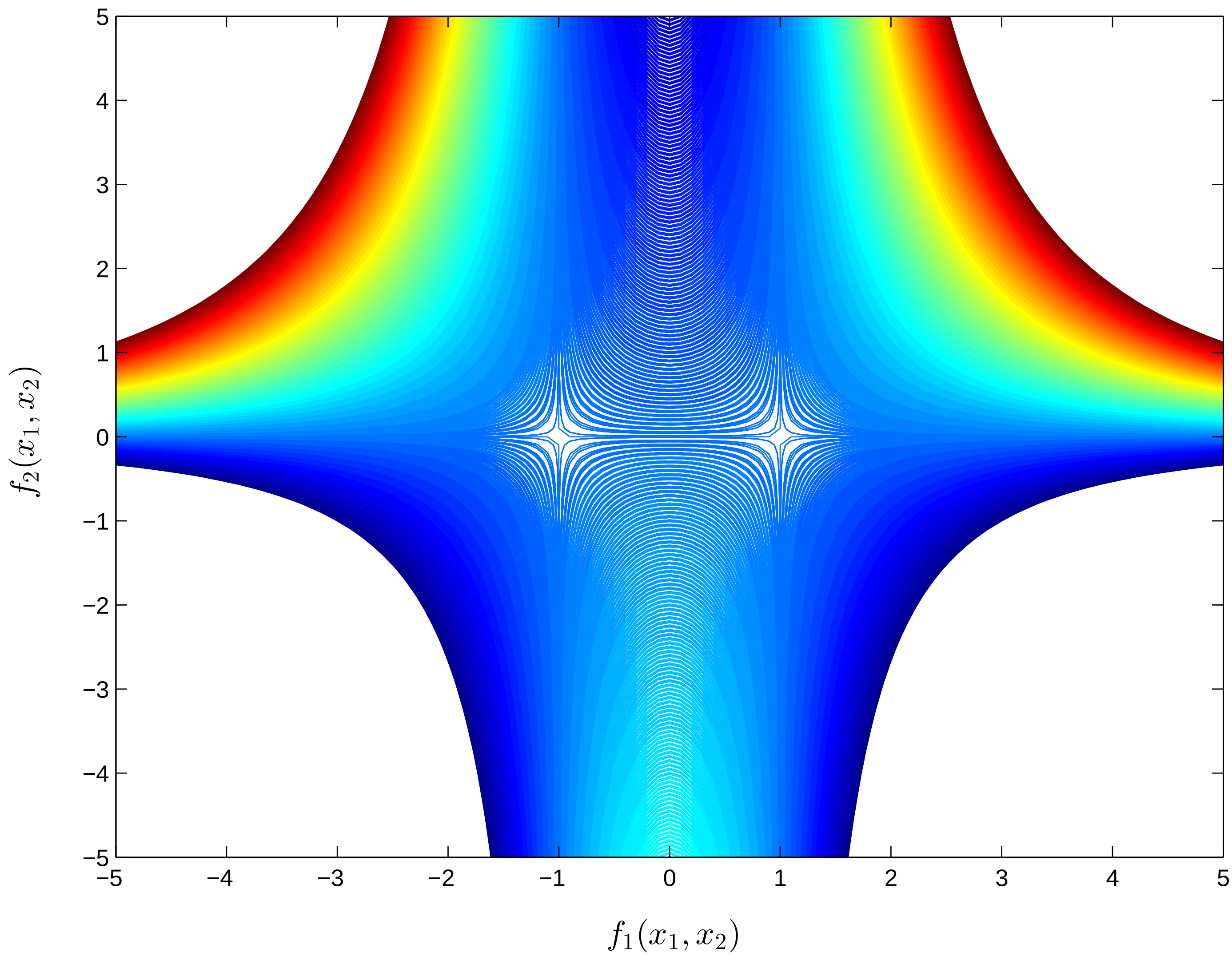
$$Df(1,\sqrt{2})=\begin{bmatrix}2\sqrt{2} & 0\\ 1 & 2\sqrt{2}\end{bmatrix}\text{ not negative definite} \Rightarrow \textit{unstable}$$

$$Df(1,-\sqrt{2})=\begin{bmatrix}-2\sqrt{2} & 0\\ 1 & -2\sqrt{2}\end{bmatrix}\text{ negative definite} \Rightarrow \textit{stable}.$$

$$Df(-1,2)=\begin{bmatrix}-4 & 0\\ 1 & 4\end{bmatrix}\text{ not negative definite} \Rightarrow \textit{unstable}$$

$$Df(-1,-2)=\begin{bmatrix}4 & 0\\ 1 & -4\end{bmatrix}\text{ not negative definite} \Rightarrow \textit{unstable}.$$

   I actually plotted the phase portraits, but I don't know what it means. You can see them on the electronic submission.


5
4
3
2
1
0
−1
−2
−3
−4
−5
$f_2(x_1,x_2)$
−5
−4
−3
−2
−1
0
1
2
3
4
5
$f_1(x_1,x_2)$