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THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5243
INTRODUCTION TO CYBER-PHYSICAL SYSTEMS

HOMEWORK # 5
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To facilitate checking, questions are color-coded blue and pertinent answers follow in regular font. The printed version however, will be in black and white.

Problem 1 — Cost of an Infinite Horizon LQR

Prove that the total cost of the CT, LTI infinite horizon LQR problem, given by:

$$\begin{aligned} \text{minimize } J &= \int_0^\infty \|y(t)\|^2\, dt \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) \\ & y(t) = Cx(t) \end{aligned}$$

is $J = x_0^\top P x_0$ where P is the solution to the steady-state Ricatti equation, given in Module 05, and $x(0)$ is the vector of initial state conditions.

Hint: Write the cost function as a quadratic cost function in terms of $x(t)$ and then relate to CARE.
Response. We can write

$$\min \quad J = \int_0^\infty y(t)^T y(t) dt = \int_0^\infty x(t)^T C^T C x(t) dt \tag{1a}$$

$$\text{Subject to} \quad \dot{x}(t) = Ax(t). \tag{1b}$$

Problem (1) is an LQR with $Q = 2CC^T$ and $B = \mathbf{0}$, $R = \mathbf{I}$ of appropriate sizes. The Ricatti equation is :

$$Q + PA + A^T P = 0. \tag{1c}$$

If we solve (1c) with $Q = CC^T$ then our optimal solution is given by $J = \frac{1}{2} x_0^T P x_0$. But now since we are solving (1c) with $Q = 2CC^T$, then the optimal cost will be $J = x_0^T P x_0$.

The proof that $J = \frac{1}{2} x_0^T P x_0$ for infinite horizon LQR is given in the linked PDF in problem 3 (http://goo.gl/CUIwPl).

-10, see solution.

Problem 2 — Infinite Horizon LQR

Compute $J = \int_0^\infty x^\top \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} x\, dt$, given that the system dynamics are given by:

$$\dot{x}(t) = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} x(t),$$

where $x(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$. You are supposed to solve the problem analytically using two different methods of your choice (CARE is one of them). You are not supposed to use any programming tool. You should also use the result from Problem 1.

Verify your solutions using MATLAB. Show your code.
Response. Method one is solving the differential equation $\dot{x}(t) = Ax$ explicitly and replacing it into the integral. To do that, we need to be able to evaluate e^{At} which we find by calculating the eigenvalues of A and writing A in the diagonal form. For the sake of brevity, only the diagonal form of A is presented:

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}. \tag{2}$$

Now we can obtain $x(t) = e^{A(t-t_0)}x(t_0)$:

$$x(t) = e^{A(t-t_0)}x(t_0) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{-2(t-t_0)} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3}$$

$$x(t)^T \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} x(t) = 2e^{-2(t-t_0)} + 2e^{-4(t-t_0)}. \tag{4}$$

We use (4) to calculate the integral, while setting $t_0 = 0$:

$$J = \int_0^\infty x(t)^T \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} x(t)dt = \int_0^\infty 2e^{-2(t-t_0)} + 2e^{-4(t-t_0)}dt = e^{2t_0} + \frac{1}{2}e^{4t_0} = \frac{3}{2}. \tag{5}$$

Method two is using CARE with $Q = 2\begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 12 \\ 12 & 8 \end{bmatrix}$, $R = \mathbf{I}_2$, $A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ be the symmetric positive semidefinite variable:

$$\begin{aligned}
&Q + PA + A^T P = \mathbf{0} \\
&\Rightarrow \begin{bmatrix} 20 & 12 \\ 12 & 8 \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\begin{bmatrix} -6P_{11} + 4P_{12} + 20 & -2P_{11} + 12 \\ -2P_{11} + 12 & -2P_{12} + 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\Rightarrow P_{11} = 6, P_{12} = 4, P_{22} = 3.
\end{aligned} \tag{6}$$

Hence, $P = \begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix}$ and the optimal cost is $\frac{1}{2}x_0^T P x_0 = \frac{1}{2}\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{2}$. The matlab code follows below:

```
A=[-3 -1; 2 0];
B=[0; 0];
Q=[20 12; 12 8];
R=1;
[K,P,e]=lqr(A,B,Q,R);

x0=[0;1];
optCost=0.5*x0.'*P*x0;
%
% P =
%
%     6     4
%     4     3

% optCost =
%
%    1.5000
```

Problem 3 — Two Point Boundary Value Problem

In this problem, we will learn about optimal control solutions for a two point boundary value problem (TPBVP) — an optimal control problem where terminal state conditions are pre-specified. You should do research on how to solve TPBVP with fixed final and initial states.

For example, you might find Example 6-1 in `http://goo.gl/CUIwPl` useful, as it includes an example on solving TPBVP. Also, read the ***LQR Variational Solution*** section in the linked PDF.

After reading the linked PDF and going through the example, find the optimal control trajectory, $u^*(t) = [u_1^*(t)\ u_2^*(t)]^\top$, that minimizes this performance index:

$$J = \frac{1}{2}\int_0^1 \|u(t)\|^2,$$

subject to:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} u,\ x(0) = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix},\ x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using MATLAB, plot your optimal control, performance index, J, and the corresponding state-trajectory.

Response. We use $\mathbf{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$ and construct the Hamiltonian:

$$H(x,u,\mathbf{p},t) = \frac{1}{2}u^T u + \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix} [\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}]$$

$$H(x,u,\mathbf{p},t) = \frac{1}{2}u^T u - 2p_2(t)x_2(t) + (\frac{1}{2}p_1(t) - \frac{1}{2}p_2(t))u_1(t) + (\frac{1}{2}p_1(t) + \frac{1}{2}p_2(t))u_2(t). \tag{7}$$

Next we use the condition that $\dot{\mathbf{p}}(t) = -(\frac{\partial H}{\partial x})^T$:

$$\dot{\mathbf{p}}(t) = \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix} = -(\frac{\partial H}{\partial x})^T = -\begin{bmatrix} 0 \\ -2p_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 2p_2(t) \end{bmatrix}. \tag{8}$$

Two differential equations are achieved in (8):

$$\dot{p}_1(t) = 0 \Rightarrow p_1(t) = c_1 \quad \text{constant.} \tag{9}$$

$$\dot{p}_2(t) = 2p_2(t) \Rightarrow \frac{\dot{p}_2(t)}{p_2(t)} = 2 \rightarrow \int \frac{dp_2}{p_2} = \int 2dt \Rightarrow ln|p_2| = 2t + \text{Const.} \Rightarrow p_2(t) = Ke^{2t} \tag{10}$$

where c_1 and K are constants to be determined later. Next, we obtain the optimal control by setting $H_u = \mathbf{0}$ as follows:

$$H_u = u^T + \begin{bmatrix} \frac{1}{2}p_1 - \frac{1}{2}p_2 & \frac{1}{2}p_1 + \frac{1}{2}p_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\Rightarrow u_1 = -\frac{1}{2}(p_1 - p_2) \quad \text{and} \quad u_2 = -\frac{1}{2}(p_1 + p_2). \tag{11}$$

We substitute (9) and (10) into (11) to obtain the optimal control:

$$u_1 = -\frac{c_1}{2} + \frac{K}{2}e^{2t} \tag{12}$$

$$u_2 = -\frac{c_1}{2} - \frac{K}{2}e^{2t}. \tag{13}$$

Next step entails using the state equations and the initial values to obtain the constants c_1 and K:

$$\dot{x}_1(t) = \frac{1}{2}u_1(t) + \frac{1}{2}u_2(t) = -\frac{c_1}{2} \Rightarrow x_1(t) = -\frac{c_1}{2}t + c_2. \tag{14}$$

$$\dot{x}_2(t) = -2x_2(t) - \frac{1}{2}u_1(t) + \frac{1}{2}u_2(t) = -2x_2(t) - \frac{1}{2}Ke^{2t} \Rightarrow x_2(t) = A_1e^{-2t} - \frac{K}{8}e^{2t} \tag{15}$$

where c_2 and A_1 are constants to be determined using the initial and terminal conditions. Note that the differential equation in (15) is solved by finding a homogenous as well as a particular solution. As mentioned earlier, we use the initial and terminal conditions to find constants c_1, c_2, K and A_1:

$$x_1(0) = c_2 = \frac{1}{2} \tag{16}$$

$$x_1(1) = -\frac{c_1}{2} + \frac{1}{2} = 0 \Rightarrow c_1 = 1 \tag{17}$$

$$x_2(0) = A_1 - \frac{K}{8} = -\frac{1}{2} \tag{18}$$

$$x_2(1) = A_1e^{-2} - \frac{K}{8}e^2 = 0 \Rightarrow A_1 = -0.5093, K = -0.0746. \tag{19}$$

Substituting these values into (12) and (13) will yield the optimal control inputs:

$$u_1(t) = -\frac{1}{2} - \frac{0.0746}{2}e^{2t} \tag{20}$$

$$u_2(t) = -\frac{1}{2} + \frac{0.0746}{2}e^{2t} \tag{21}$$

Also $J(t)$ which is the performance index at any time t is calculated by:

$$J(t) = \frac{1}{2}\int_0^t ||u(t')||^2 dt' = \frac{t}{2} + (6.9565e^{4t} - 6.9565)10^{-4}$$

.

The optimal control inputs are shown in Fig. 1, performance index is depcited in Fig. 2 and the optimal state trajectories is plotted in Fig. 3.

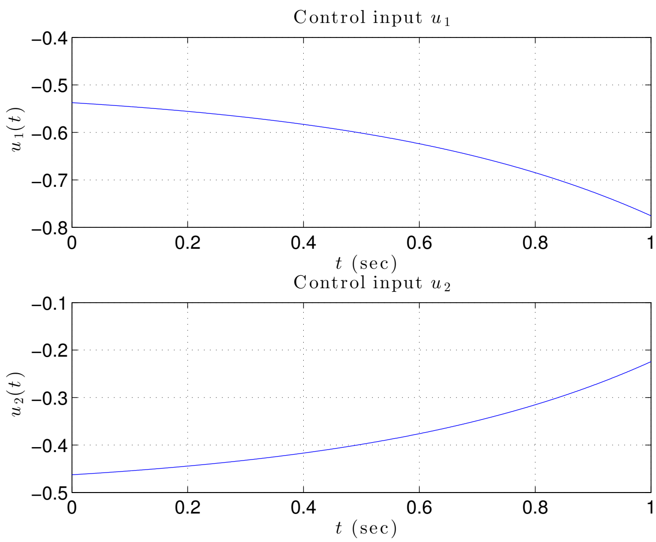

Figure 1: Optimal control inputs for problem 3.

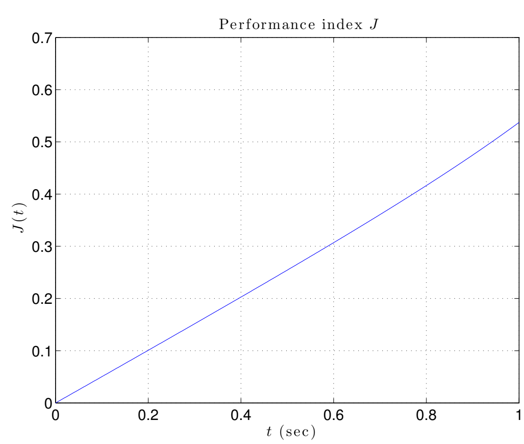

Figure 2: Performance index J for problem 3.

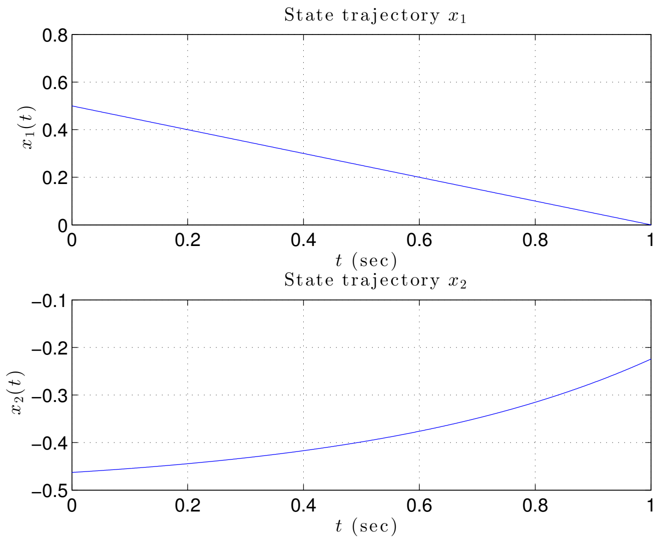

Figure 3: State trajectroy for problem 3.

Problem 4 — Optimal Control for a Nonlinear System

Using the HJB-equation, and a candidate quadratic function $V(t, x)$, find the optimal control action that minimizes this performance index:

$$J = (x(1))^2 + \int_0^1 x^2(t)u^2(t)\, dt,$$

where the dynamics of the nonlinear system are given by:

$$\dot{x}(t) = x(t)u(t), \;\; x(0) = 1.$$

You'll have to follow the following steps:

1. Construct the Hamiltonian.
2. Obtain the optimal control.
3. Apply property 1 of any value function, i.e., $\frac{\partial V}{\partial x} = \lambda^*(x, t)$.
4. Substitute this optimal control and multiplier into the HJB equation.
5. Formulate a candidate quadratic value function.
6. Use this candidate in the HJB and obtain an ODE that relates to the value function.
7. Obtain the optimal control.
8. Finally, plot the optimal control, cost function, and the state trajectory (you can either use the `ode` solver on MATLAB).

Response.

1. $H(x, u, \lambda, t) = x^2u^2 + \lambda x u$.
2. $\frac{\partial H}{\partial u} = 0 \Rightarrow 2x^2u + \lambda x = 0 \Rightarrow u^* = -\frac{\lambda}{2x}$.
3. $\frac{\partial V}{\partial x} = \lambda^*(x, t)$.

4. $\min_u H(x,u,\lambda^*,t) = x^2(-\frac{\lambda^*}{2x})^2 + \lambda^* x(-\frac{\lambda^*}{2x}) = -\frac{\lambda^{*2}}{4} = -\frac{\partial V}{\partial t}$.

5. $V(x,t) = P(t)x^2 \Rightarrow \frac{\partial V}{\partial x} = 2P(t)x$ and $\frac{\partial V}{\partial t} = \dot{P}(t)x^2$.

6. $-\dot{P}(t)x^2 = -\frac{\partial V}{\partial t} = \min_u H(x,u,\lambda^*,t) = -\frac{\lambda^{*2}}{4}$ and $\lambda^*(x,t) = \frac{\partial V}{\partial x} = 2P(t)x$. Therefore we obtain:

$$\dot{P}(t)x^2 = \frac{4P^2(t)x^2}{4} \Rightarrow \dot{P}(t)x^2 = P^2(t)x^2 \Rightarrow \dot{P}(t) = P^2(t). \tag{22}$$

   Notice that also since $V(x,1) = P(1)x(1)^2 = x(1)^2$, we need to have $P(1) = 1$.

7. We solve the ODE in (22):

$$\frac{dP}{dt} = P^2 \Rightarrow \frac{dP}{P^2} = dt \Rightarrow \int \frac{1}{P^2}dP = \int dt \Rightarrow -\frac{1}{P(t)} = t + C \Rightarrow P(t) = -\frac{1}{t+C} \tag{23}$$

   where C is a constant that needs to be determined. Using $P(1) = 1 \Rightarrow -\frac{1}{1+C} = 1 \Rightarrow C = -2$. Therefore $P(t) = -\frac{1}{t-2}$. Replace $P(t)$ in $V(x,t) = P(t)x^2$ to obtain

$$V(x,t) = -\frac{1}{t-2}x^2(t),$$

$$\lambda = -\frac{2}{t-2}x(t),$$

   and

$$u = -\frac{\lambda}{2x} = \frac{-\frac{2}{t-2}x(t)}{2x(t)} = -\frac{1}{t-2}.$$

   this should be -1/(2-t)

   Finally, we use the state equation to solve for $x(t)$:

$$\dot{x}(t) = x(t)u(t) = -\frac{x(t)}{t-2} \Rightarrow \frac{\dot{x}}{x} = -\frac{1}{t-2} \Rightarrow ln|x(t)| = -ln|2-t| + \text{Const} \rightarrow x(t) = \frac{K}{2-t} \tag{24}$$

   where K is a constant that needs to be determined using $x(0)$. Also notice that, the differential equation has a solution since $t \in [0,1]$ and hence $t < 2$. Using the initial conditions

$$x(0) = 1 \Rightarrow \frac{K}{2-0} = 1 \Rightarrow K = 2.$$

   This yields the explicit equation for the state trajectory:

$$x(t) = -\frac{2}{t-2}.$$

   At this point we have everything we need to plot the results.

8. The plots are given in Fig. 4

Problem 5 — Discrete LQR Solution + Lagrangian

The discrete dynamics of an LTI system is given by:

$$x_{k+1} = Ax_k + Bu_k, \ \ k = 0, 1, \dots, N-1.$$

Consider the optimal control problem of finding optimal control sequence, $u_0^*, \dots, u_{N-1}^*$, given:

- Specific initial and final conditions: x_0 and x_N are fixed, and
- Cost index: $J = \frac{1}{2}\sum_{k=0}^{N-1} u_k^\top R u_k$

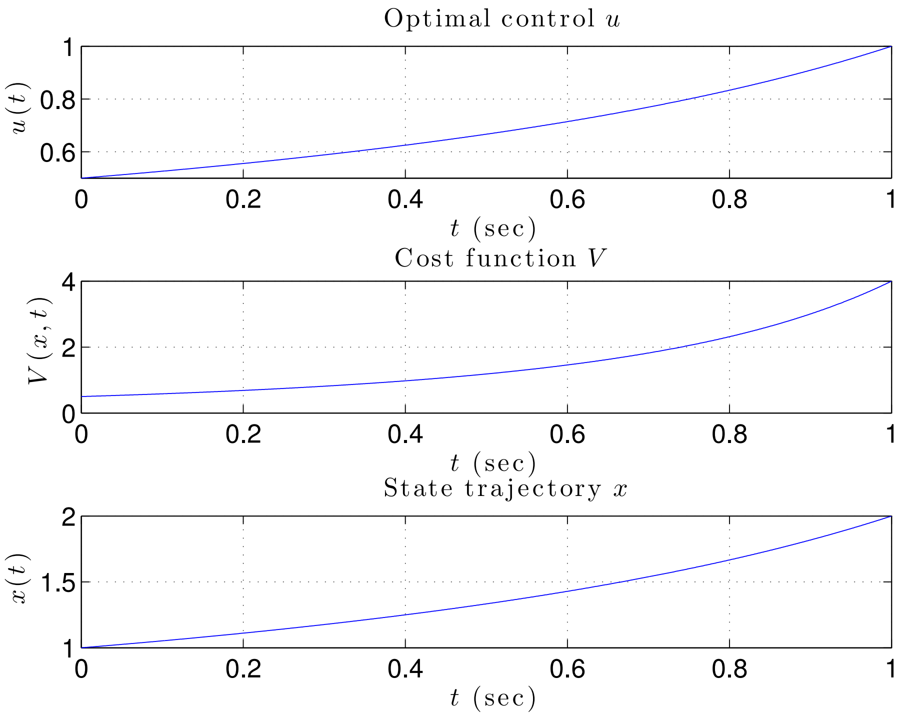

Figure 4: Optimal control, cost function and the state trajectory for problem 4.

The objective of this problem is to transform the optimal control problem to a quadratic, static optimization problem subject to linear equality constraints.

Answer the following questions:

1. Write the cost function J as a quadratic cost function in $u = \begin{bmatrix} u_0 & u_1 & \dots & u_{N-1} \end{bmatrix}^\top$, where you should determine the quadratic cost-matrix—it should be diagonal.
2. Given an initial and final fixed states, write the dynamics of the system as $A_u u = b_u$, where A_u and b_u should be determined in terms of A, B, x_0, x_{N-1}.
3. Formulate the optimal control problem as a quadratic program with linear equality constraints.
4. Construct the Lagrangian of the transformed optimization problem.
5. What is the optimal u^*? You have to solve a KKT-like problem for multipliers and control.

Response. Let $u = \begin{bmatrix} u_0^T & u_1^T & \dots & u_{N-1}^T \end{bmatrix}^T$.

1. We can write $J = \frac{1}{2} u^T \mathbf{R} u$ where $\mathbf{R} = \begin{bmatrix} R_k & 0 & \dots & 0 \\ 0 & R_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_k \end{bmatrix}$ which is a block diagonal matrix.

2.

$$
\begin{aligned}
x_1 &= Ax_0 + Bu_0 \\
x_2 &= Ax_1 + Bu_1 = A(Ax_0 + Bu_0) + Bu_1 = A^2x_0 + ABu_0 + Bu_1 \\
x_3 &= Ax_2 + Bu_2 = A(A^2x_0 + ABu_0 + Bu_1) + Bu_2 = A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2 \\
&\vdots \\
x_N &= A^N x_0 + A^{N-1}Bu_0 + A^{N-2}Bu_1 + \ldots + ABu_{N-2} + Bu_{N-1}
\end{aligned}
$$

Which yields to the following system of equations:

$$\left[A^{N-1}B, A^{N-2}B, \ldots, AB, B\right] u = x_N - A^N x_0. \tag{25}$$

Therefore we will have $A_u = \left[A^{N-1}B, A^{N-2}B, \ldots, AB, B\right]$ and $b_u = x_N - A^N x_0$.

3. The new formulation will be a linear constrained quadratic program:

$$\min_u \quad \frac{1}{2}u^T \mathbf{R} u \tag{26}$$
$$\text{subject to } A_u u = b_u \tag{27}$$

For ease of representation, let $A_u \in \mathbb{R}^{m \times n}$, $b_u \in \mathbb{R}^m$, $u \in \mathbb{R}^n$.

4. Introduce $\lambda \in \mathbb{R}^m$. The Lagrangian is then :

$$L(u, \lambda) = \frac{1}{2}u^T \mathbf{R} u + \lambda^T (A_u u - b_u). \tag{28}$$

5. We start by writing out the KKT conditions:

$$\nabla_u L(u, \lambda) = \mathbf{R}u + A_u^T \lambda^* = 0 \Rightarrow u^* = -\mathbf{R}^{-1}(A_u^T)\lambda^*. \tag{29}$$

Notice that $\mathbf{R}$ is a square matrix and its inverse hopefully exists (specially since its block-diagonal). Next, we find the λ^* by substituting (29) into (27):

$$A_u u = b_u = A_u(-\mathbf{R}^{-1})(A_u^T)\lambda^* = b_u \Rightarrow \lambda^* = \left[A_u(-\mathbf{R}^{-1})(A_u^T)\right]^{-1} b_u \tag{30}$$

Here, also notice that $\left[A_u(-\mathbf{R}^{-1})(A_u^T)\right] \in \mathbb{R}^{m \times m}$ and its inverse potentially exists. Finally plugging in (30) into (29) yields the optimal u^*:

$$u^* = -\mathbf{R}^{-1}(A_u^T)\lambda^* = -\mathbf{R}^{-1}(A_u^T)\left[A_u(-\mathbf{R}^{-1})(A_u^T)\right]^{-1} b_u. \tag{31}$$

Problem 6 — Principle of Optimality and DP

Consider the following discrete-time LQR problem:

$$
\begin{aligned}
\text{minimize} \quad & J && = (x_2 - 10)^2 + \frac{1}{2}\sum_{k=0}^{1}(x_k^2 + u_k^2) \\
\text{subject to} \quad & x_{k+1} && = 2x_k - 3u_k \\
& x(0) && = 4.
\end{aligned}
$$

The final state can be anything, i.e., it is free, not fixed. The objective of this problem is to solve the above optimal control problem by invoking the Principle of Optimality and Dynamic Programming, similar to the example on Slide 13 of Module 05.

Obtain the optimal control sequence and the corresponding state trajectories.
Response. We start by calculating $J_2^*(x_2) = (x_2 - 10)^2$ and knowing that $x_2 = 2x_1 - 3u_1$:

$$J_1^*(x_1) = \min_{u_1}\{\frac{1}{2}(x_1^2 + u_1^2) + (x_2 - 10)^2\} = \min_{u_1}\{\frac{1}{2}(x_1^2 + u_1^2) + (2x_1 - 3u_1 - 10)^2\} \tag{32}$$

Setting $\frac{\partial J_1}{\partial u_1} = 0$ yields:

$$\frac{\partial J_1(x_1)}{\partial u_1} = u_1 - 6(2x_1 - 3u_1 - 10) \Rightarrow u_1^* = \frac{-60 + 12x_1}{19}. \tag{33}$$

Plugging in (33) into (32) results in:

$$J_1^*(x_1) = \frac{27}{38}x_1^2 - \frac{40}{19}x_1 + \frac{100}{19}. \tag{34}$$

Next we obtain

$$J_0^*(x_0) = \min_{u_0}\{\frac{1}{2}x_0^2 + \frac{1}{2}u_0^2 + J_1^*(x_1)\} = \min_{u_0}\{\frac{1}{2}x_0^2 + \frac{1}{2}u_0^2 + \frac{27}{38}x_1^2 - \frac{40}{19}x_1 + \frac{100}{19}\} \tag{35}$$

for which we replace x_1 with its equivalent $x_1 = 2x_0 - 3u_0$ to obtain:

$$J_0^*(x_0) = \min_{u_0}\{\frac{1}{2}x_0^2 + \frac{1}{2}u_0^2 + \frac{27}{38}(2x_0 - 3u_0)^2 - \frac{40}{19}(2x_0 - 3u_0) + \frac{100}{19}\}. \tag{36}$$

To find optimal u_0^* we need to take the derivative of (36) and set it equal to zero:

$$\frac{\partial J_0(x_0)}{u_0} = u_0 - 3 \times \frac{2 \times 27}{38}(2x_0 - 3u_0) - \frac{40}{19}(-3) = 0 \tag{37}$$

resulting in $u_0^* = \frac{8.5263x_0 - 6.3158}{13.7895}$. The problem gives $x_0 = 4$ which we substitute to obtain $u_0 = 2.0153$. Then, $x_1 = 2x_0 - 3u_0 = 2 \times 4 - 3 \times 2.0153 = 1.9541$. Using (33), $u_1^* = \frac{-60+12\times x_1}{19} = -1.9237$. Finally $x_2 = 2x_1 - 3u_1 = 9.6793$.