Your Name:

Your Signature:

- **Exam duration:** 3 hours.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If I find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place $\boxed{\text{a box around your final answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 30 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	30	
2	25	
3	35	
4	30	
5	20	
6	15	
7	25	
8	20	
Total	200	

1. (30 total points) You are given the following LTI dynamical system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

(a) (5 points) What are the modes/eigenvalues of A? Is the system stable?

(b) (5 points) Is the above system controllable or not? Justify your answer.

(c) (5 points) Is the above system observable or not? Justify your answer.

(d) (5 points) Obtain the unobservable subspace of the system—if it exists.

(e) (5 points) Is there a state feedback controller $u(t) = -Kx(t)$ such that $A - BK$ has eigenvalues $\{-2, -1, -3\}$? If yes, find this state feedback gain K. Justify why if your answer is no.

(f) (5 points) Is there a state observer such that $A - LC$ has eigenvalues $\{-4, -1, -2\}$? If yes, find this state feedback gain L. Justify why if your answer is no.

2. (25 total points) The following LTV system is given:

$$\dot{x}(t) = A(t)x(t) = \begin{bmatrix} -\alpha + \beta\cos(t) & -3 \\ 3 & -\alpha + \beta\cos(t) \end{bmatrix} x(t).$$

   (a) (10 points) First, find the matrix exponential of this matrix for any real a and b:

$$A_1 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

(b) (15 points) Use the answer in the previous part to find the state-transition matrix of $A(t)$.

3. (35 total points) You are given the following SISO system:

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$

   (a) (20 points) Design an observer-based controller (i.e., $u(t) = -K\hat{x}(t)$) for the above system such that the desired eigenvalues for the closed loop system are all at $\lambda_{cl} = \{-2, -3\}$ for both the controller and the observer.

   First, you'll have to check if the system is controllable and observable (or detectable and stabilizable).

(b) (5 points) Draw a block diagram representation of the overall system with the observer based controller, including the values for the gains K and L that you have designed.

(c) (10 points) Write a MATLAB code to simulate the observer-based controller you designed above.

4. (30 total points) The nonlinear, spinning body dynamics of a satellite can be written as

$$\begin{aligned}
\dot{\omega}_1(t) &= \frac{I_2 - I_3}{I_1}\omega_2(t)\omega_3(t) + \frac{1}{I_1}\tau_1(t) \\
\dot{\omega}_2(t) &= \frac{I_3 - I_1}{I_2}\omega_3(t)\omega_1(t) + \frac{1}{I_2}\tau_2(t) \\
\dot{\omega}_3(t) &= \frac{I_1 - I_2}{I_3}\omega_1(t)\omega_2(t) + \frac{1}{I_3}\tau_3(t)
\end{aligned}$$

where $I_{1,2,3}$ are the moments of inertia about principal axes (and are constants); $\omega_{1,2,3}$ are the angular velocities about principal axes; $\tau_{1,2,3}$ are the torques and control inputs about principal axes.

(a) (5 points) Consider that the system states are the three angular velocities and that the control inputs are the three torques. What is a trivial equilibrium point (i.e., control inputs and state equilibrium points) of this system?

(b) (10 points) Obtain the linearized representation of the system around the trivial equilibrium point.

(c) (5 points) Determine the stability of the system around the equilibrium point.

(d) (10 points) Is the linearized system controllable? Stabilizable? Justify your answer. You should give two solutions to this problem: the first based on the properties of controllability we discussed in class, and another solution based on the physical interpretation of the linearized dynamics.

5. (20 total points) *[You're halfway through the exam. You're getting there. Remember, look at the glass half-full, because emptiness is harder to quantify.]*

   Consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t).$$

   (a) (20 points) Prove that the above system is controllable if the controllability matrix is full-rank.

6. (15 total points) Consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = x_{t_0}.$$

   (a) (15 points) Prove that the closed-form to the above differential equation for any time-varying control input is given by:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

   Note that to prove that a certain function is a solution to any ODE, you have to prove that the initial conditions hold, and that the analytic solution is true for all $t > t_0$.

   Hint — Leibniz Differentiation Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,\mathrm{d}x\right) = \int_{a(\theta)}^{b(\theta)} \partial_\theta f(x,\theta)\,\mathrm{d}x + f\big(b(\theta),\theta\big)\cdot b'(\theta) - f\big(a(\theta),\theta\big)\cdot a'(\theta)$$

7. (25 total points) Consider the following DT LTI system

$$x(k+1) = Ax(k) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} x(k), \; y(k) = Cx(k) = \begin{bmatrix} -1 & 1 \end{bmatrix} x(k).$$

   (a) (5 points) Is A nilpotent? Of what order?

(b) (10 points) Suppose $y(0) = 1$ and $y(1) = 0$. Can we **uniquely find** $x(0)$? If yes, find it. If not, explain why you cannot.

(c) (10 points) Suppose $y(1) = 1$ and $y(2) = 0$. Can we **uniquely find** $x(0)$? If yes, find it. If not, explain why you cannot.

8. (20 total points) *[This is the final question of the exam. This painful experience is about to end. I told you to look at the glass half full.]*

   Consider the following system

$$
\begin{aligned}
\dot{x}(t) &= TJT^{-1}x(t) + Bu(t) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u(t) \\
y(t) &= Cx(t) \\
&= \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} x(t).
\end{aligned}
$$

   (a) (10 points) Obtain necessary conditions on the entries of B such that only λ_1 is controllable. This means λ_2 is simply not controllable.

   I'm not giving you the eigenvectors because they're beautiful—they're given for a purpose.

(b) (10 points) Obtain necessary conditions on the entries of C such that only λ_2 is observable. This means λ_1 is simply not observable.

9. (20 total points) **This is a bonus question and is relatively trickier than other questions.**
   (a) (20 points) For the LTI model

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with Gramian $W(t,t_0)$, prove that the STM for

$$\begin{bmatrix} A & BB^\top \\ 0 & -A^\top \end{bmatrix}$$

is

$$\begin{bmatrix} \phi_A(t,t_0) & -\phi_A(t,t_0)W(t,t_0) \\ 0 & \phi_A^\top(t_0,t) \end{bmatrix}.$$