Who Are I: Intrapersonal Conflicts and Self Control

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Prologue: Sirens and Odysseus

Sirens and Odysseus

Curious to hear the Sirens' songs but mindful of the danger...

Figure: By John William Waterhouse (1891)

Time Consistency

Optimal Control and Dynamic Programming

- Optimal control

$$\begin{array}{ll} \text{Minimise} & J(u(\cdot)) = \int_0^T f(t, x(t), u(t))dt + h(x(T)) \\ \text{subject to} & \dot{x}(t) = b(t, x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n \end{array} \tag{1}$$

- *Dynamic programming*
- A family of problems

$$\begin{array}{ll} \text{Minimise} & J(s, y; u(\cdot)) = \int_s^T f(t, x(t), u(t))dt + h(x(T)) \\ \text{subject to} & \dot{x}(t) = b(t, x(t), u(t)), \quad x(s) = y \end{array} \tag{2}$$

Bellman's principle of optimality and HJB equation

- *Value function* $V(s,y) = \inf_{u(\cdot)} J(s,y;u(\cdot))$
- Bellman's principle of optimality (BPO)

$$V(s,y) = \inf_{u(\cdot)|_{[s,s']}} \left[\int_s^{s'} f(t,x(t),u(t))dt + V(s',x(s')) \right], \ \forall 0 \le s \le s' \le T$$

- V solves HJB (classical or viscosity)

$$-v_t + \sup_u H(t,x,u,-v_x) = 0, \quad v(T,x) = h(x)$$

where *Hamiltonian* $H(t,x,u,p) = p \cdot b(t,x,u) - f(t,x,u)$

- *Verification theorem*

$$u^*(t,x) = \text{argmax}_u H(t,x,u,-v_x(t,x))$$

Time Consistency Illustrated

Time consistency (necessary condition of BPO): $u^*(\cdot)$ optimal on $[s, T]$ with initial $(s, y) \implies u^*(\cdot)|_{[s',T]}$ optimal on $[s', T]$ with initial $(s', x^*(s'))$ for $s' > s$

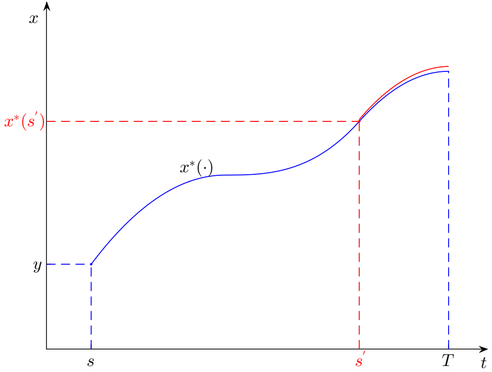

Figure: Time Consistency

Optimal Control with Discounting

- Change objective to

$$J(u(\cdot)) = \int_0^T e^{-rt} f(t, x(t), u(t))dt + e^{-rT} h(x(T))$$

- The (s, y) problem is

$$\begin{aligned} J(s, y; u(\cdot)) &= \int_s^T e^{-r(t-s)} f(t, x(t), u(t))dt + e^{-r(T-s)} h(x(T)) \\ &= e^{rs} \left[\int_s^T e^{-rt} f(t, x(t), u(t))dt + e^{-rT} h(x(T)) \right] \end{aligned}$$

- BPO

$$e^{-rs} V(s, y) = \inf_{u(\cdot)|_{[s,s']}} \left[\int_s^{s'} e^{-rt} f(t, x(t), u(t))dt + e^{-rs'} V(s', x(s')) \right]$$

- So it is still time consistent
- HJB and verification

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- *Stationarity axiom*: Rate of discount is constant over time

Stochastic Control

Minimise $\quad J(u(\cdot)) = \mathbb{E}\left[\int_0^T f(t, x(t), u(t))dt + h(x(T))\right]$

subject to $\quad dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \quad x(0) = x_0$

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  - A careful definition of "admissible (open-loop) control" (*weak formulation*)
  - Tower rule of conditional expectation: $\mathbb{E}[\xi|\mathcal{F}_s] = \mathbb{E}\left[\mathbb{E}(\xi|\mathcal{F}'_s)|\mathcal{F}_s\right], \ \forall s \le s'$

Stochastic BPO

- *Stochastic BPO*

$$V(s,y) = \inf_{u(\cdot)|_{[s,s']}} \mathbb{E}_{s,y}\left[\int_s^{s'} f(t,x(t),u(t))dt + V(s',x(s'))\right],\ 0 \le s \le s' \le T$$

  where $\mathbb{E}_{s,y} := \mathbb{E}(\cdot|x(s)=y)$

- Time consistency holds
- HJB and verification

Optimal Stopping

Minimise $J(\tau) = \mathbb{E}[h(x(\tau))]$

subject to $dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \quad x(0) = x_0$

- Time consistency holds
- Variational inequalities

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  - Regime change between (US) republican and democratic administrations
- There are far more time inconsistent problems than consistent ones (Strotz 1956, Kydland and Prescott 1977)

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- Probability weighting

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- Stationarity of time preference is violated
- *Present bias*: we promise ourselves to be patient in the distant future, but submit ourselves to the desire for **instant** pleasure

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- $r(t) \equiv r$ for $\rho(t) = e^{-rt}$

Optimal Control with Decreasing Impatience

- Problem

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Continuous-Time Markowitz Problem

- Problem

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- BPO fails!
- Time inconsistent (Z. and Li 2000, Basak and Chabakauri 2010)

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  - This time: B was more popular (insurance)
- Exaggeration of extremely **small** probabilities

Probability Weighting - Yaari's Dual Theory

- Preference on random payoff $X \geq 0$ represented by (Yaari 1987)

$$V(X) := \int_0^\infty w(\mathbb{P}(X > x))dx = \int_0^\infty x w'(1 - F_X(x)) dF_X(x)$$

where $w : [0,1] \to [0,1]$, $\uparrow$, $w(0) = 0$, $w(1) = 1$ and F_X is CDF of X

- Overweighting both very good and very bad outcomes when $w(\cdot)$ is inverse-S shaped
- *Choquet expectation*: $\tilde{\mathbb{E}}[X] = \int_0^\infty w(\mathbb{P}(X > x))dx$ - **nonlinear expectation**

Probability Weighting Function

- Kahneman and Tversky (1992):

$$w(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}},$$

$0 \le \delta \le 1.$

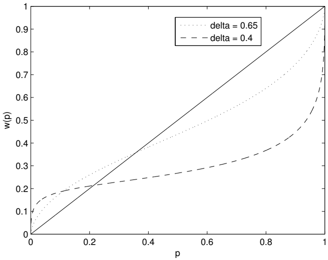

Figure: Inverse-S Shaped Probability Weighting Function $(\delta = 0.65, \delta = 0.4)$

Stochastic Control under Probability Weighting

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- No dynamic programming

Time-Inconsistent Optimal Stopping

- Optimal stopping with hyperbolic discounting (O'Donoghue and Rabin 1999, Grenadier and Wang 2007, Ebert, Wei and Z. 2017)
- Optimal stopping under probability weighting (Xu and Z. 2013, Ebert and Strack 2015, Huang, Nguyen-Huu and Z. 2017)
- Casino gambling models (Barberis 2012, He, Hu, Obłój and Z. 2014, 2015)

Who Are I: Intrapersonal Conflicts and Self Control

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- Time inconsistency gives rise to *self-control* problem: Phelps and Pollak (1968), O'Donoghue and Rabin (1999)

Dealing with Time Inconsistency

Three Types of Agents under Time Inconsistency

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- Descriptive, rather than prescriptive
- The three types are **identical** in a time-consistent problem

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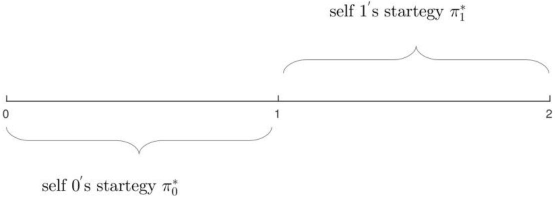
- Resulting strategy is called an *equilibrium*: no "self" will be better off by deviating from the equilibrium
- Existence and uniqueness: extremely challenging problems!!!

Idea Explained via Two-Period Model

- Idea best explained in a two-period model
- Objective is to maximise $J(i, X_2)$, $i = 0, 1$
- Self 1 solves a one-period optimisation problem, with optimal strategy π_1^* and optimal final state $X_2^* = f(x_1, \rho_{12})$
- Self 0 maximises $J(0, f(X_1, \rho_{12}))$ subject to budget constraint, to get strategy π_0^*
- (π_0^*, π_1^*) is an equilibrium strategy



Extension to Continuous Time

- Self s forms an alliance with all the sleves in $[s, s+\varepsilon]$ and lets $\varepsilon \to 0$
- Given a control u^*, for any $s \in [0, T)$, $\varepsilon > 0$ and $v \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^l)$, define

$$u^{s,\varepsilon,v}(t) = u^*(t) + v\mathbf{1}_{t\in[s,s+\varepsilon)}, \quad t \in [s, T].$$

- Let u^* be given and x^* be the corresponding state process
- Assuming the objective is to minimise, u^* is called an *equilibrium* if

$$\liminf_{\varepsilon \downarrow 0} \frac{J(s, x^*(s); u^{s,\varepsilon,v}) - J(s, x^*(s); u^*)}{\varepsilon} \geq 0,$$

  for any $s \in [0, T)$ and $v \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^l)$
- Karp (2004), Ekeland and Lazrak (2006), Björk and Murgoci (2009), Yong (2011), Hu, Jin and Z. (2012), Björk, Murgoci and Z. (2014)

A Portfolio Choice Model with RDU Preference

Rank-Dependent Utility Theory

- Rank-dependent utility theory (RDUT): Quiggin (1982), Schmeidler (1989)
- Preference dictated by an RDUT **pair** (u, w)

$$\int_0^{+\infty} w(\mathbb{P}(u(X) > y))dy + \int_{-\infty}^{0} \left(w(\mathbb{P}(u(X) > y)) - 1\right) dy$$

- Two components
  - A concave (outcome) utility function u: individuals dislike mean-preserving spread
  - A (usually assumed) inverse-S shaped (probability) weighting function w: individuals overweight tails

A Portfolio Choice Model under Rank-Dependent Utility

$$\begin{array}{ll} \text{Maximise} & J(s,y;\pi(\cdot)) \\ \text{subject to} & dX(t) = \pi(t)^\top \mu(t)dt + \pi(t)^\top \sigma(t)dW(t), \ \ X(s) = y \end{array}$$

where

$$J(s,y;\pi(\cdot)) = \int_0^{+\infty} w(s, \mathbb{P}_s(u(X(T)) > y))dy + \int_{-\infty}^0 (w(s, \mathbb{P}_s(u(X(T)) > y)) - 1)dy$$

with $w(s,\cdot)$ being the probability weighting applied at time s, $u(\cdot)$ the (outcome) utility function, and $\mathbb{P}_s$ the conditional probability given $\mathcal{F}_s$, which includes the information $X(s) = y$

If There Is No Probability Weighting...

- If $w(s,p) \equiv p$ then the RDUT model reduces to the (time-consistent) *Merton problem*
- Define the *deflator process*

$$\rho(t) \triangleq \exp\left(-\frac{1}{2}\int_0^t |\theta(s)|^2 ds - \int_0^t \theta(s)^\top dW(s)\right)$$

  where $\theta(t) = \sigma(t)^{-1}\mu(t)$
- Then the optimal portfolio is the replicating portfolio of the claim

$$X(T) = I(\lambda\rho(T))$$

  where $I = (u')^{-1}$
- $\rho(T)$: *pricing kernel* or *stochastic discounting factor* or *state price density*
- Optimal terminal wealth is *anti-comonotonic* w.r.t. pricing kernel, if u is concave
- Important implications in asset pricing, market equilibria, etc.

A Function, An ODE, and A Process

- Define a function

$$h(t,x) \stackrel{\triangle}{=} \mathbb{E}\left[w_p'(t,N(\xi))e^{x\xi}\right], \;\; t \in [0,T], \; x \in \mathbb{R},$$

where $w_p'(t,p) = \frac{\partial}{\partial p}w(t,p)$, ξ is a standard normal random variable, and N is CDF of ξ

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- Define an ODE

$$\begin{cases} \Lambda'(t) = -\theta(t)^2\left(\frac{h(t,\sqrt{\Lambda(t)})}{h'(t,\sqrt{\Lambda(t)})}\right)^2\Lambda(t), \;\; t\in[0,T), \\ \Lambda(T) = 0 \end{cases}$$

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- Define a process

$$\bar{\rho}(t) \triangleq \exp\left(-\frac{1}{2}\int_0^t |\lambda(s)\theta(s)|^2 ds - \int_0^t \lambda(s)\theta(s)^\top dW(s)\right)$$

where $\lambda(t):=\sqrt{-\Lambda'(t)/|\theta(t)|^2}$ with $\Lambda(\cdot)$ being a positive solution of the ODE

Main Result

Theorem. (Hu, Jin and Z. 2016) Under some technical conditions, and assume that the ODE admits a solution $\Lambda(\cdot)$ with $\Lambda(t) > 0\ \forall t \in [0, T)$, and that the following inequality holds for any $c \in \mathbb{R}$:

$$\int_{-\infty}^{+\infty} w_p'\left(t, N\left(\frac{c-g(x)}{\sqrt{\Lambda(t)}}\right)\right) N'\left(\frac{c-g(x)}{\sqrt{\Lambda(t)}}\right)\left(g''(x)+\frac{c-g(x)}{\Lambda(t)}g'(x)^2\right) du(x) \geq 0, \quad a.e. t \in [0, T)$$

where $g(x) = -\ln u'(x)$. Then the portfolio replicating the terminal wealth

$$X(T) = I\left(e^{\frac{1}{2}\Lambda(0)}\bar{\rho}(T)\right)$$

is an equilibrium strategy.

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- The final wealth is anti-comonotonic with $\bar{\rho}(T)$
- Asset pricing implication: the pricing kernel should probably be $\bar{\rho}(T)$?

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- Definition of equilibrium strategy:

$$\limsup_{\varepsilon \downarrow 0} \frac{J(t, X(t); \pi^{t,\varepsilon,k}) - J(t, X(t); \pi)}{\varepsilon} \leq 0 \quad \forall (t, k)$$

Idea of Proof (Cont'd)

- After an enormous amount of calculations, we deduce

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& \limsup_{\varepsilon \downarrow 0} \frac{J(t, X(t); \pi^{t,\varepsilon,k}) - J(t, X(t); \pi)}{\varepsilon} \\
\leq \; & -|\sigma(t)^\top k|^2 \frac{1}{2\sqrt{\Lambda(t)}} \int_{-\infty}^{\infty} w_p'(t, N(Y(0))) N'(Y(0)) \left(g''(m(0)) + g'(m(0))^2 \frac{Y(0)}{\sqrt{\Lambda(t)}} \right) dy \\
& + \theta(t)^\top \sigma(t)^\top k \frac{1}{\sqrt{\Lambda(t)}} \int_{-\infty}^{\infty} w_p'(t, N(Y(0))) N'(Y(0)) g'(m(0)) \left(1 - \frac{Y(0)}{\sqrt{\Lambda(t)}} \lambda(t) \right) dy
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- However, the right hand side is quadratic in k ...
- ... hence we have an equality which is the ODE, and an inequality which is the condition in the theorem

Recent Works on Continuous-Time Equilibria

- Deterministic consumption with non-exponential discounting (Ekeland and Lazrak 2006)
- Merton problem with non-exponential discounting (Ekeland and Pirvu 2008)
- Stochastic consumption/investment with decreasing impatience (Wei and Z. 2015)
- General time-inconsistent stochastic control (Björk and Murgoci 2009, Yong 2012)
- Continuous-time Markowitz problem (Björk, Murgoci and Z. 2014, Dai, Jin, Kou and Xu 2017)
- Optimal stopping with decreasing impatience (Huang and Nguyen-Huu 2016, Ebert, Wei and Z. 2017)
- Rank-dependent utility maximisation (Hu, Jin and Z. 2017)
- Optimal stopping under probability weighting (Huang, Nguyen-Huu and Z. 2017)

Epilogue:
Rules Rather Than Discretion

Rules, Discretion, and Compromises

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  - Odysseus: got himself bound to the mast

New Research Opportunities

- Time-inconsistency: largely unexplored in control and mathematical finance
- New opportunities begging for innovation research